APPLICATION OF LAPLACE TRANSFORMS IN THE SOLUTION OF INTEGRAL EQUATIONS

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ABSTRACT

In the present work certain integral equation involing the H-function [1, p.49] as kernel has been transformed, by introducing new Gamma function factors into the integrand by means of operator L into another integral equation with a symmetrical Fourier kernel introduced by Fox [4] and the solution is then immediate. L and L^{-1} denote the Lapace transform and its inverse.

1. INTRODUCCION. Fox [2], [3] and Verma [8] have shown that L and L^{-1} operators, when acting upon Mellin type integrals, have the power, to annihilate or to introduce new Camma function factors into the integrand. Here we have employed the powers of L of introducing new Gamma function factors into the integrand for transforming the given integral equation into another integral equation with a symmetrical Fourier kernel given by Fox [4] and the solution is then immediate. Since this symmetrical Fourier kernel is a generalisation of a large variety, of functions that occur frequently in problem may be of general interest.

Integral equations of the type

$$(1.1) \quad \int_0^\infty H^{0,\rho}_{2\rho,\nu}\left(xu \,\middle|\, \begin{matrix} (a_i, e_i)_{1,2\rho} \\ (b_i, c_i)_{1,\nu} \end{matrix} \right) f(u)\, du = \phi(x) ,$$

where Φ is given and f is the function to be found, will be considered here.

In (1.1),

$$H^{0,\rho}_{2\rho,\nu}\left(x \; \begin{matrix} (a_i ; e_i)_{1;2\rho} \\ (b_i ; c_i)_{1;\nu} \end{matrix} \right)$$

(1.2)

$$= (2\pi i)^{-1} \int_T \frac{\prod_1^{\rho} \Gamma(a_i - e_i s)}{\prod^{\nu} \Gamma(b_i + c_i - c_i s) \prod^{\rho} \Gamma(a_i - e_i + e_i s)} x^{-s}\, ds ,$$

and the following assumptions are made:

(i) $c_i > 0$, $i = 1, \ldots, q$; $e_j > 0$, $j = 1, \ldots, p$;

(ii) all the poles of the integrand of (1.2) are simple;

(iii) the contour T is a straight line parallel to the imaginary axis in the s $(= \sigma + \tau)$ plane and the poles of $\Gamma(a_i - e_i s)$, $i = 1, \ldots, p$, lie to the right of it;

(iv) $D = \sum_1^{\nu} c_i - 2 \sum_1^{\rho} e_i > 0$;

(v) $\mathrm{Re}(a_j) > e_j/2D$, $j = 1, \ldots, p$; $\mathrm{Re}(b_j) \geqslant c_j/2D$, $j = 1, \ldots, q$.

We shall write (1.2) in the form:

(1.3)

$$H^{o;\rho}_{2\rho,\nu}\left(x \;\begin{matrix} (a_i\,;\,e_i)\ 1\ ;\ 2\rho \\ (b_i\,,\,c_i)\ 1\ ,\ \nu \end{matrix} \right) = (2\pi i)^{-1} \int_T M_{\nu,2\rho}(s)\, x^{-s}\, ds ,$$

where

$$(1.4) \quad M_{\nu,2\rho}(s) = \prod_1^{\rho} \Gamma(a_i - e_i s) \left\{ \prod_1^{\nu} \Gamma(b_i + c_i - c_i s) \prod_1^{\rho} \Gamma(a_i - e_i + e_i s) \right\}^{-1}$$

For the sake of brevity, we shall write H (x) for the left hand side of (1.3).

from (1.3), it follows that

$$(1.5) \quad M \left[H^{0,\rho}_{2\rho,\nu}\left(x \begin{matrix} (a_i\,;\,e_i)\ 1\ ;\ 2\rho \\ (b_i\,,\,c_i)\ 1\ ,\ \nu \end{matrix} \right) \right] = M_{\nu,2\rho}(s) ;$$

where M denotes the Mellin transform.

We now estimate the asymptotic behaviour of $M_{\nu,2\rho}(s)$, $s = \sigma + i\tau$, σ and τ real when $|\tau|$ large. For large s the asymptotic expansion of Gamma function [7] is given by

(1.6) $\log \Gamma(s+a) = (s+a-\tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log(2\pi) + 0(s^{-1})$,

where $|\arg s| <$, and 0 is the usual order symbol. To find the asymptotic expansion of $M_{\nu,2\rho}(s)$ for large $|\tau|$, we replace Gamma functions involving $-s$ into those containing $+s$, with the help of the relation.

(1.7) $\Gamma(z)\,\Gamma(1-z) = \pi \operatorname{cosec} \pi z.$

Then using (1.6) and the assumptions, (i), (iv) and (v), one can obtain.

(1.8) $M_{\nu,2\rho}(s) = |\tau|^{D(\sigma-\frac{1}{2})} \exp\{ i\tau (D \log|\tau| - B) \}\{ Q + O(|\tau|^{-1}) \}$,

where Q is a constant which may have one value when τ is large and positive and another value when τ is large and negative. From (1.8), it follows that if $6 < \tfrac{1}{2}$ then the integral of (1.3) is uniformly convergent with respect to x. It can be extended for the cast $\sigma = \tfrac{1}{2}$.

In this section we find the asymptotic expansion of

$H^{0,\rho}_{2\rho,\nu}\left(x \begin{matrix} (a_i \;\; e_i)_{1,2\rho} \\ (b_i, c_i)_{1,\nu} \end{matrix} \right)$ when x is large and positive, with error term $0(x^{-\sigma_0})$, where 0 is the order symbol.

THEOREM 1. If,

(i) $D = \sum_1^{\nu} c_i - 2 \sum_1^{\rho} e_i > 0$;

(ii) $\alpha = \prod_1^{\nu} (c_i|D)^{c_i|D} \Big/ \prod_1^{\rho} (e_i|D)^{2e_i|D}$;

(iii) $\beta = \alpha^D$; $K = \sum_1^{\nu} (b_i + c_i) - 2 \sum_1^{\rho} a_i$;

(iv) $c_i > 0, i = 1, \ldots, q$; $e_i > 0, i = 1, \ldots, p$;

(v) x is real and positive;

(vi) for given $\sigma_0 > \tfrac{1}{2}$, N denotes the greatest integer in $\{D(2\sigma_0 - 1) + 3\}/2$ and M_j denotes the greatest positive integer in $\{e_i\sigma_0 - \mathrm{Re}(a_i)\}$, $i = 1, \ldots, p$;

(vii) $\text{Re}\,(b_i) > 0\,, i=1\,, \ldots, q\,;$

$\text{Re}\,(a_i) > e_i/2\,, i=1\,, \ldots, p\,;$

(viii) $q-p$ is an odd positive integer;

then

$$H^{0,\rho}_{2\rho,\nu}\left(x \,\middle|\, \begin{matrix} (a_i \;\; e_i)\; 1, 2\rho \\ (b_i\,, c_i)\; 1\,, \nu \end{matrix} \right)$$

(1.9)

$$= (x|\beta)^{(1-D)|2D} \sum_{j=0}^{N} \nu_j\, (x|\beta)^{-j|D} \quad \sin\left\{ \left(K-j+\frac{1-D}{2}\right) (\pi/2) + (x|\beta)^{1/D} \right\}$$

$$+ \sum_{j=1}^{\rho} x^{-a_j|e_j} \left(\Lambda_j + B_j\, x^{1|e_j} + \,.\, + U_j\, x^{-M_j|e_j} \right)$$

$$+ 0 \left(x^{-\sigma_0} \right),$$

where ν_j, $j=0\,, 1, \ldots, N\,;$

$\Lambda_j\,, B_j\,, \ldots, U_j\,;\; j=1, \ldots, p\,;$

are constants depending on the parameters

$a_i\,, e_i\,;\; i=1, \ldots, p\,;$
$b_i\,, c_i\,;\; i=1, \ldots, q\,;$
but are indipendent of x

If $q-p$ is an even positive integer (instead of condition (viii) the in the above expression we have to replace sine by cosine. The proof of this theorem is on similar lines as in Fox [4, p. 417].

2. THE LAPLACE AND MELLIN TRANSFORMS.

The Laplace transform of $\Phi(x)$ is defined by

(2.1) $$L\{\Phi(x)\} = \int_0^{\infty} e^{-xt}\, \Phi(x)\, dx = \psi(t).$$

If $M\,[h(u)] = H(s)$ and $M\,[f(u)] = F(s)$, then the Mellin-Parseval Theorem states that

(2.) $$\int_0^{\infty} h(u)\, f(u)\, du = (2\pi\,)^{-1} \int_C H(s)\, F(1-s)\, ds\,,$$

where S is a suitable contour in the s-plane.

3. THE SOLUTION OF (1.1) AS AN INTEGRAL EQUATION.

THEAREM 2. If,

(i) $\mathrm{Re}(b_i) > -c_i/2$, $i=1,\ldots,q$; $c_i>0$, $i=1,\ldots,q$;

(ii) $f(x) \in L_2(0,\infty)$;

(iii) $s^{D(s-\frac{1}{2})} F(1-s) \in L(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$;

(iv) $F(1-s) \in L(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$;

(v) $y^{-\frac{1}{2}} f(y) \in L(0,\infty)$ where $f(y)$ is of bounded variation near the point $y=x$, then the solution of (1.1), as an integral equation for $f(u)$ is

$$(3.1)\quad f(x) = \int_0^\infty H^{\nu,\rho}_{2\rho,2\nu}\left(xu \begin{matrix} (a_i,e_i)\ 1,2\rho \\ (b_i,c_i)\ 1,2\nu \end{matrix} \right) \times$$

$$\times [\, t^{b1} L\{ \ldots [\, t^{b}\nu - 2\, L\{ \ldots [u^{\nu-1}\, L\{\tau^{\nu-1-1}\, [t^{b}\nu\, L\{$$

$$L\{ u^{b}\nu^{-1}\, \Phi(u^{-c}\nu) \}]_{t=(1/\tau)} (c_{\nu-1}/c_\nu) \}] \ldots \}]_{t=u} {}^{1/c_1}\, du .$$

PROOF. Firstly, we apply (2.2) to the left-hand side of (1.1). For large positive u and $x>0$, the asymptotic expansion of $H^{0,\rho}_{2\rho,\nu}(x)$ discussed in (1.9) gives us

(3.2)

$$H^{0,\rho}_{2\rho,\nu}\left(xu \begin{matrix} (a_i,e_i)\ 1,2\rho \\ (b_i,c_i)\ 1,\nu \end{matrix} \right)$$

$$= (xu/\beta)^{(1-D)/2D} \sum_{j=0}^{N} \nu_j\, (xu|\beta)^{-j|D} \sin\{ (K-j+(1-D)/2)\pi/2 + (\tfrac{xu}{\beta})^{1|D} \}$$

$$+ \sum_{j=1}^{\rho} (xu)^{-a_j|e_j} \{ \Lambda_j + B_j (xu)^{-1|e_j} + \ldots + U_j (xu)^{-M_j|e_j} \}$$

$$+ 0\left(u^{-\sigma_0} \right) ,$$

where ν_j ; $j=0,1,\ldots,N$;

$A_j, B_j, \ldots\ldots, U_j$; $j=1, \ldots\ldots, p$;

are constants depending on the parameters

a_i, e_i ; $i=1, \ldots\ldots, p$,

b_i, c_i ; $i=1, \ldots\ldots, q$;

but are independent of x,u.

It follows from (3.2) that $H^{0,p}_{2p,v}(xu) \in L_2(0,\infty)$. This result, combined with condition (ii), $f(u) \in L_2(0,\infty)$, justifies the use of Theorem 72,p.95[6], with $k=\frac{1}{2}$, so that we can apply (2.2) to the left-hand side of (1.1). Using (1.4), we obtain

$$(3.3) \quad \Phi(x)=(2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{v,2p}(s)\, x^{-s}\, F(1-s)\, ds\ , \ (x>0)\ ,$$

where $M_{v,2p}(s)x^{-s}$ and $F(s)$ are Mellin transforms of $H^{0,p}_{2p,v}(xu)$ and $f(u)$ respectively, and the contour in the s-plane is the straight line $s=\frac{1}{2}+i\tau$, τ varies from $-\infty$ t0 ∞.

In this part of the proof, we shall try to introduce q new Gamma function factors into the integrand of (3.3) by using the power of L operator of introducing new Gamma function factors. In the first instance, we introduce the q−th new Gamma function factor $\Gamma(b_v+c_v s)$ in the integrand by using the technique of operator L. Then, in similar manner, we can introduce the remaining (q−1) factors, namely, $\Gamma(b_1+c_1 s), \ldots\ldots\ldots\ldots, \Gamma(b_{v-1}+c_{v-1}s)$, and thus, we shall arrive at a result with symmetrical Fourier kernel and then solution is immediate.

Now we use operator L to introduce the Gamma function factor $\Gamma(b_v+c_v s)$, $c_v>0$, $Re(b_v)>-c_v/2$, into the integrand of (3.3).

Considering $\Phi(x^{-c}v)$ we are led to the result:

$$(3.4) \quad \Phi(x^{-c_v})=(2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M_{v,2p}(s)\, x^{c_v s}\, F(1-s)\, ds$$

Using operator L, we find

$$L\{x^{b_v-1}\,\Phi(x^{-c_v})\}$$

(3.5)

$$= \int_0^{\infty} e^{-tx} x^{b_\nu - 1} \{ (2\pi i)^{-1} \int_{1/2 - i\infty}^{1/2 + i\infty} M_{\nu, 2p}(s)\, x^{c_\nu s} F(1-s)\, ds \}\, dx$$

Now using (1.8), we can write for large $|\tau|$, $s = 1/2 + i\tau$,

$$(3.6) \quad M_{\nu, 2p}(s)\, F(1-s) = s^{D(s-1/2)} F(1-s) \{ Q_1 + O(s^{-1}) \},$$

where Q_1 is a constant which may have one value when τ is large and positive and another value when τ is large and negative.

Since $s = 1/2 + i\tau$, the real power of x in (3.5) is Re $(b_\nu) + 1/2 c_\nu - 1$, which by condition (i), exceeds -1. Also, by (3.6) and condition (iii), the terms in s in (3.5) belong to $L(1/2 - i\infty, 1/2 + i\infty)$. Hence, the integrand in (3.5) is an absolutely convergent double integral and we can integrate with respectto x. The result, thus, found is

$$L\{ x^{b_\nu - 1} \Phi(x^{-c_\nu}) \}$$

(3.7)

$$= (2\pi i)^{-1} \int_{1/2 - i\infty}^{1/2 + i\infty} M_{\nu, 2p}(s)\, \Gamma(b_\nu + c_\nu s)\, t^{-b_\nu - c_\nu s} F(1-s)\, ds .$$

Similarly, we introduce the Gamma function factor $\Gamma(b_{\nu-1} + c_{\nu-1} s)$ in the integrand. Let us take $t = (1/\tau)^{(c_{\nu-1}|c_\nu)}$, in (3.7). Again, using the operator L and justifying the change in the order of integration, we get

$$L \left\{ \tau^{b_{\nu-1}-1} \left[t^{b_\nu} L \left\{ x^{b_\nu - 1} \Phi(x^{-c_\nu}) \right\} \right]_{t=(1|\tau)^{(c_{\nu-1}|c_\nu)}} \right\}$$

(3.8)

$$= (2\pi i)^{-1} \int_{1/2 - i\infty}^{1/2 + i\infty} M_{\nu, 2p}(s)\, \Gamma(b_{\nu-1} + c_{\nu-1} s)\, \Gamma(b_\nu + c_\nu s)\, x^{-b_{\nu-1} - c_{\nu-1} s} F(1-s)\, ds .$$

Hence, by means of $(q-2)$ successions of L operator we can arrive at the result:

$$[t^{b_1} L \{ \ldots . [x^{b_{\nu-1}} L \{ \tau^{b_{\nu-1}-1} [t^{b_\nu} L \{ x^{b_\nu - 1} \Phi(x^{-c_\nu}) \}]$$

(3.9) $$\Big]_{t=(1/\tau)} (c_{\nu-1}|c_\nu) \Big\} \Big] \ldots \Big\} \Big]_{t=(x^{1/c_1})}$$

$$=(2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_{2\nu,\,2\rho}(s)\, x^{-s}\, F(1-s)\, ds ,$$

where

(3.10) $$M_{2\nu,\,2\rho}(s)=\prod_{1}^{\nu} \Gamma(b_i+c_is) \prod_{1}^{\nu} \Gamma(a_i-e_is) \times$$

$$\times \Big\{ \prod_{1}^{\nu} \Gamma(b_i+c_i-c_is) \prod_{1}^{\rho} \Gamma(a_i-e_i+e_is) \Big\}^{-1}$$

If we write left-hand side of (3.9) equal to $\Psi(x)$, then (3.9) takes the following concise form:

(3.11) $$\psi(x)=(2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_{2\nu,\,2\rho}(s)\, x^{-s}\, F(1-s)\, ds$$

or

$$\psi(x)=(2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M\left[H^{\nu,\,\rho}_{2\rho,\,2\nu}\left(u \,\middle|\, \begin{matrix} (a_i, e_i)\ 1, 2\rho \\ (b_i, c_i)\ 1, 2\nu \end{matrix} \right) \right] x^{-s}\, F(1-s)\, ds$$

in the light of (1.5).

On using [5, p. 391, eqn. (12)] to the right hand side if (3.12), it can be expressed by an integral involving the product of

$H^{\nu,\,\rho}_{2\rho,\,2\nu}\left(x u \,\middle|\, \begin{matrix} (a_i, e_i)\ 1, 2\rho \\ (b_i, c_i)\ 1, 2\nu \end{matrix} \right)$ and $f(u)$. The result, is given by the relation.

(3.13) $$\int_0^\infty H^{\nu,\,\rho}_{2\rho,\,2\nu}\left(xu \,\middle|\, \begin{matrix} (a_i, e_i)\ 1, 2\rho \\ (b_i, c_i)\ 1, 2\nu \end{matrix} \right) f(u)\, du = \psi(x) :$$

Sinee $H^{\nu,\,\rho}_{2\rho,\,2\nu}\left(xu \,\middle|\, \begin{matrix} (a_i, e_i)\ 1, 2\rho \\ (b_i, c_i)\ 1, 2\nu \end{matrix} \right)$ is a symmetrical Fourier kernel introduced by Fox [4, p. 408, eqn. (52)], we are thus led formally to the solution as:

$$(3.14) \quad f(x) = \int_0^\infty H^{\nu,\rho}_{2\rho,2\nu}\left(xu \begin{matrix} (a_i, e_i)\ 1, 2\rho \\ (b_i, c_i)\ 1, 2\nu \end{matrix} \right) \psi(u)\, du$$

$$(3.15) \quad = \int_0^\infty H^{\nu,\rho}_{2\rho,2\nu}\left(xu \begin{matrix} (a_i, e_i)\ 1, 2\rho \\ (b_i, c_i)\ 1, 2\nu \end{matrix} \right) \times$$

$$\times [t^{b_1} L\{\ldots L\{\tau^{b_{\nu-1}}[t^{b_\nu} L\{u^{b_\nu-1}\Phi(u^{-c_\nu})\}]$$

$$]_{t=(1/\tau)}(c_{\nu-1}|c_\nu)\}]\ldots\}]_{t=u^{1/c_1}}\, du\ .$$

4. PARTICULAR CASES.

(i) By taking $e_i=1$, $i=1,\ldots, p$; $c_i=1$, $i=1,\ldots, q$, in Theorem 2, the integral equation

Theorem 2, the integral equation

$$(4.1) \quad \int_0^\infty G^{o,\rho}_{2\rho,\nu}\left(xu \begin{matrix} (a_i)\ 1, 2\rho \\ (b_i)\ 1, \nu \end{matrix} \right) f(u)\, du = \Phi(x)\ , \ (x>0)\ ,$$

has the solution

$$(4.2) \quad f(x) = \int_0^\infty G^{\nu,\rho}_{2\rho,2\nu}\left(xu \begin{matrix} (a_i)\ 1, 2\rho \\ (b_i)\ 1, \nu \end{matrix} \right) \times$$

$$\times [t^{b_1} L\{\ldots [u^{b_{\nu-1}} L\{\tau^{b_\nu}\ 1\ [t^{b_\nu} L\{u^{b_\nu-1}\Phi(u^{-1})\}]$$

$$]_{t=(1/\tau)}\quad \}]\ldots\}]_{t=(u)}\quad du\ :$$

where $G^{m,n}_{p,q}(x)$ is Meijer's G-function [1].

(ii) With $e_i = 1$, $i = 1, \ldots, p$; $c_i = 1$, $i = 1, \ldots, q$; $q = 1$, $p = 0$, $b_1 = \nu$ in Theorem 2, and using [1, p.216, (3)], the integral equation

$$(4.3) \quad \int_0^\infty G_{0,1}^{0;0}(xu \mid -\nu)\, f(u)\, du = \Phi(x), \quad (x>0),$$

has the salution

$$(4.14) \quad f(x) = \int_0^\infty J_{2\nu}[2(xu)^{1/2}]\,[t^\nu L\{\Phi(u^{-1})\}]\, du,$$

where $G_{p,v}^{m;n}(x)$ is Meijers G-function [1, p. 207].

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