A NOTE ON INTEGRATION OF LOMMEL'S FUNCTION

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1. The integral equation

(1.1) $$\phi(p)=p\int_0^\infty e^{-px}f(x)dx,\ R(p)>o$$

represen the classical Laplace transformation of one variable and the functions $\phi(p)$ and $f(x)$ related by (1.1), are said to be operationally related to each other. $\phi(p)$ is called the image and $f(x)$ the original.

Symbolically we can write

(1.2) $$\phi(p)\doteqdot F(x)\ \text{or}\ f(x)\doteqdot\phi(p),$$

and the symbol $\doteqdot$ i called 'operational'.

The object of the present note is to evaluate integrals involving Lommel's Function (see [1], page 372) by using operational calculus.

2. *The Main Theorem:* suppose

(i) $\phi(p)\doteqdot h(x)k(x)$,
(ii) $ph(\sqrt{p})\doteqdot g(x)$,
(iii) $pg(p)\doteqdot f(x)$,
(iv) $\psi(p,y)\doteqdot\dfrac{k(x)}{x^2+y}$

then

(2.1) $$\phi(p)=\int_0^\infty f(y)\psi(p,y)dy,$$

provided a change in the order of integration is permissible.
Proof: From (i) and (ii) we have

(2.2) $$\phi(p)=p\int_0^\infty e^{-px}h(x)\ k(x)\ dx$$

and

$$(2.3) \qquad h(\sqrt{p}) = \int_0^\infty e^{-px}\, g(x)\, dx.$$

Now using (2.3) in (2.2), we get

$$(2.4) \qquad \phi(p) = p\int_0^\infty e^{-px} k(x) \left[\int_0^\infty e^{-x^2 t}\, g(t)\, dt \right] dx.$$

By interpreting with the help of (iii), we have

$$\phi(p) = p\int_0^\infty e^{-px}\, k(x) \left[\int_0^\infty e^{-x^2 t} \left\{ \int_0^\infty e^{-yt}\, f(y)\, dy \right\} dt \right] dx.$$

On changing the order of integration, it follows

$$(2.5) \qquad \phi(p) = p\int_0^\infty e^{-px}\, k(x) \left[\int_0^\infty f(y) \left\{ \int_0^\infty e^{-(x^2+y)t}\, dt \right\} dy \right] dx$$

$$= p\int_0^\infty e^{-px}\, k(x) \left[\int_0^\infty \frac{f(y)}{x^2+y}\, dy \right] dx$$

$$= \int_0^\infty f(y) \left[p\int_0^\infty e^{-px} \frac{k(x)}{x^2+y}\, dx \right] dy$$

$$\phi(p) = \int_0^\infty f(y)\, \psi(p,y)\, dy, \quad \text{by(IV)}.$$

This completes the proof of the theorem.

3. *Integrals of Lommel's functions.*

Suppose $h(x) = \exp(x^2/4) D_{-\mu}(x)$ and $k(x) = x^\nu$

$$h(x)k(x) = x^\nu \exp(x^2/4) D_{-\mu}(x) \doteqdot p^{\mu-\nu} \sum_{r=o}^{\infty} \frac{(\mu)_{2r}}{r!}$$

$$\Gamma(\nu - 2r - \mu + 1) \left(\frac{-p^2}{2}\right)^r = \phi(p)$$

(See [1], result (4), page 210)

$$ph(\sqrt{p}) = pe^{p/4}D_{-\mu}(\sqrt{p}) \doteqdot \frac{x^{\mu/2-1}(x+\tfrac{1}{2})^{-\mu/2-\frac{1}{2}}}{2^{(\mu+1)/2}\Gamma(\mu/2)} \equiv g(x)$$

(See [1], result (20), page 139)

$$pg(p) = \frac{1}{2^{\frac{\mu+1}{2}}\Gamma(\mu/2)} \cdot \frac{p^{\mu}/2}{(p+\tfrac{1}{2})^{-\frac{\mu+1}{2}}} \doteqdot \frac{\Gamma\frac{(1-\mu)}{2}e^{-x/4}}{2^{\mu-\frac{1}{2}}\Gamma(\mu/2)\pi}.$$

$$.[D_{\mu-1}(-\sqrt{x}) - D_{\mu-1}(\sqrt{x})] \equiv f(x)$$

(See [1], result (4), page 210).

$$\frac{K(x)}{x^2+y} = \frac{x^{\nu}}{x^2+y} \doteqdot \pi \operatorname{cosec}[(\nu+1)\pi] y^{\frac{\nu-1}{2}}\, p\, V_{\nu+1}(2\sqrt{yp},o) \equiv \psi(p,y)$$

(See [1], result (9), page 138).

Hence from (2.1), we get

$$(3.1) \qquad \int_0^{\infty} y^{\frac{\nu-1}{2}} \exp(-\frac{y}{4}) \ [D_{\mu-1}(\sqrt{y}) - D_{\mu-1}(\sqrt{y})] V_{\nu+1}(2\sqrt{yp},o)dy$$

$$= \frac{\Gamma(\frac{\mu}{2})2^{\mu+\frac{1}{2}}p^{\mu-\nu-1}}{\Gamma(\frac{1-\mu}{2}) \operatorname{cosec}[(\nu+1)\pi]} \sum_{r=o}^{\infty} \frac{(\mu)2r}{r!} \Gamma(\nu-2r-\mu+1) \ (-\frac{p^2}{2})^r,$$

$2 > \mathrm{Re}(\nu) > -1$

Similarly if we consider the pairs $\{h(x), k(x)\}$ as:

$$h(x) = \begin{cases} 2\sqrt{\pi}x^{-2}\log x \\ x^{\nu-2}k_{\nu}(2\sqrt{ax}) \\ \dfrac{1}{x(x+a)} \end{cases} \text{ y } k(x) = \begin{cases} x^{n+1} \\ x^{\mu-\nu+2} \\ \\ x^{\nu-1} \end{cases}$$

and follow the technique given in (3.1), then we arrive at the following results:

$$(3.2)\qquad \int_0^\infty y^{\frac{n-1}{2}} \log y \; V_{n+2}(2\sqrt{yp},o)\, dy = \frac{2\Gamma(n+1)}{p^{n+1} \operatorname{cosec}[(n+2)\pi]}$$

$[\psi(n+1)-\log p]$, $|\operatorname{Re}(n)|<1$.

(3.3)

$$\int_0^\infty Y^{\frac{\mu-1}{2}} J_\nu(2\sqrt{ay})\, V_{\mu-\nu+3}(2\sqrt{yp},o)dy = -\frac{2\sin(\mu\pi)\sin[\mu-\nu+3)\pi]}{\pi \sin[(\mu+\nu)\pi]}$$

$$\frac{\Gamma(\mu-\nu+1)}{(p^2-4a)^{\frac{\mu+1}{2}}} Q_\mu^\nu \left(\frac{p}{\sqrt{(p^2-4a)}}\right);$$

$\operatorname{Re}(p)>2a$, $\operatorname{Re}(\mu+\nu)>1$, $o<\operatorname{Re}(\nu)<{}^7/_2$, $\operatorname{Re}(\nu-\mu)>o$.

(3.4)

$$\int_0^\infty \frac{Y^{\frac{\nu-1}{2}}}{y+a^2} V_{\nu+2}(2\sqrt{yp},o)dy = \frac{\Gamma(1+\nu)\, a^{\frac{\nu-3}{2}} \exp\left(\frac{ap}{2}\right)}{\operatorname{cosec}[(\nu+2)\pi]\, p^{\frac{\nu+1}{2}}}$$

$$W_{-\frac{\nu+1}{2},\frac{\nu}{2}}(ap),$$

Reference $(\nu)\ |<1$.

1. Erde'lyi, Magnus, Oberhettinger, Tricomi; Tables of Integral Transforms. Vol. 1 (1954). Bateman Project. McGraw-Hill book company, inc.