A NOTE ON FRACTIONAL INTEGRATION

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1. We call

$$(1.1) \quad g(y;\mu) = \frac{1}{\Gamma(\mu)} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$$

The Riemann-Liouville fractional integral of order μ, and

$$(1.2) \quad H_v\{f(x);y\} = \int_0^\infty f(x)\,H_v\,(xy)\,\sqrt{xy}\,dx$$

the H-transform of order v, of f(x).

The object of this note is to obtain a theorem relating (1.1) and (1.2) and utilize it to deduce the fractional integral.

2. Theorem 1. Let

(i) $g(y;\mu)$ be the Riemann-Liouville fractional integral of order μ of $f(x)$,

(ii) $\Phi(\delta)$ be the H-transform of $y^{v+1/2}\,g(y^2;\mu)$ of order v,

(iii) Provided the integrals involved exist and be absolutely integrable,

then

$$(2.1) \quad \delta^{\mu-1/2}\,\Phi(\delta) = \frac{(-2)^{\mu-1}}{\Gamma(\mu)} \int_0^y x^{\frac{\mu+v}{2}} H_{\mu+v}\,(\delta\sqrt{x})\,f(x)\,dx,$$

$$R(v) > -3/2,\ R(\mu) > o.$$

Proof. We have from (i)

$$g(y;\mu) = \frac{1}{\Gamma(\mu)} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx.$$

On multiplying both sides by $y^{1/2} H_v(\delta\sqrt{y})$ and integrating between the limits (o, ∞), it follows

(2.2)

$$\int_0^\infty y^{v/2} H_v(\delta\sqrt{y})\, g(y;\mu)\,dy = \frac{1}{\Gamma(\mu)} \int_0^\infty y^{1/2} H_v(\delta\sqrt{y})\,dy \int_0^\infty f(x)\,(y-x)^{\mu-1}dx.$$

Now changing the order of integration on the right, which is justifiable due to absolute convergence mentioned in conditions of theorem one, we have

(2.3)

$$\int_0^\infty y^{v/2} H_v(\delta\sqrt{y})\, g(y;\mu)\,dy = \frac{1}{\Gamma(\mu)} \int_0^\infty f(x)\,dx \int_0^\infty y^{v/2} H_v(\delta\sqrt{y})\,(y-x)^{\mu-1}dy.$$

Evaluating the inner integral on the right hand side, with the help of the known result ([1], result 88, p. 199), and simplifying on both sides we get the required result (2.1). Hence (2.1) is proven.

3. Example. Suppose $f(x) = x^{-v/2} J_v(a\sqrt{x})$.

Therefore $$g(y;\mu) = \frac{2^{2-v} a^{-\mu} y^{\frac{\mu-v}{2}} S_{\mu+v-1,\mu-v}(a\sqrt{y})}{\Gamma(\mu)\,\Gamma(v)}$$

and $$H_v\,[y^{+v^{1/2}} g(y^2;\mu);\, \delta] = \frac{2^{2\mu+1} a^{-2\mu-3/2}}{\Gamma(\mu)\,\Gamma(v)\,\Gamma(1-\mu)\,\Gamma(1-v)}$$

$$G^{2,3}_{3,3}\left(\frac{\delta^2}{a^2} \middle| \begin{matrix} 3/4+v/2,\ 1/4-v/2-\mu,\ 1/4+v/2-\mu \\ 3/4+v/2,\ 1/4-v/2-\mu,\ 1/4+v/2 \end{matrix} \right)$$

(See [1], pages 194 and 170).

Hence from (2.1), we get

(3.1)

$$\int_{o}^{y} X^{\frac{\mu+v}{2}} H_{\mu+v}(\delta\sqrt{x})\, J_v(a\sqrt{x})\, dx = (-1)^{1-\mu} \frac{2^{\mu+2}\,\delta^{\mu-1/2}\, a^{-2\mu-3/2}}{\Gamma(v)\,\Gamma(1-v)\,\Gamma(1-\mu)}$$

$$G^{2,3}_{3,3}\left(\frac{\delta^2}{a^2}\,\begin{matrix} 3/4+v/2,\ 1/4-v/2-\mu,\ 1/4+v/2-\mu \\ 3/4+v/2,\ 1/4-v/2-\mu,\ 1/4+v/2 \end{matrix}\right),$$

$$o<R(v)<1,\ R(2\mu+v)<3/2.$$

REFERENCES

[1] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Tables of Integral Transforms. Vol. 2, McGraw-Hill Book Company, Inc. (1954).

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