A NOTE ON n th ORDER DIFFERENTIAL INEQUALITIES

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Summary. The object of this note is to add some new n th order diferential inequalities to the vast literature on differential inequialities which can be convenientrly used in the analysis of a class of n the order differential equations.

1. **Intrudiction.** During the last several years substantial number of papers have been published dealing with differential inequalities and their applications to a variety of mathematical problems of physical interest (see , [2] - [10] and the references therein). Relatively, few attempts have been made to establish n th order differential enequialities which are easily applicable to a systematic study of n the order differential equiations. However, some recent developments in the theory of n th order differential equiations have become of compelling interest in the study of such models without reducing them to a system of n equations of the first order. Our aim in this note is to establish some new n th order differential inequalities which are easily applicable to study the behavior of n th order differential equiation model considered by Z. Opial in [4] , without reducing it to a system of n equations of the first order.

2. **Main Results.** In this section we establish our main results on some fundamental n th order differential inequialities which can be used as handy tools in the analysis of a class of n th order differential equiations.

A useful n th order differential inequality is embodied in the following theorem.

THEOREM 1. Let $y(t), y'(t), \ldots, y^{(n)}(t)$, **and** $a(t)$ **be realvalued nonnegative continuous functions defined onf** $I \doteq [0, \infty)$ **except that** $y^{(n-1)}(t)$ **is positive for all** $t \in I$**, for which the inequality**

$$(1) \qquad y^{(n)}(t) \leq a(t) y^{(n-1)}(t)\left[y(t) + y'(t) + \ldots + y^{(n-1)}(t)\right],$$

holds for all t $\in$ I. If

$$1-\left[y(o)+y'(o)+\ldots+y^{(n-1)}(o)\right]\int_{o}^{t} e^{s}\, a(s)ds > o,$$

for all $\in$ I, where

$$(2)\quad y^{(n)}(t) \leq a(t)\,Q(t)\,y^{(n-1)}(o)\exp\left(\int_{o}^{t} a(s)\;\;Q(s)\,ds\right),$$

for all t $\in$ I. where

$$(3)\quad Q(t)=\frac{e^{t}\left[y(o)+y'(o)+\ldots+y^{(n-1)}(o)\right]}{1-\left[y(o)+y'(o)+\ldots+y^{(n-1)}(o)\right]\int_{o}^{t} e^{s}\, a(s)ds},$$

for all t $\in$ I.

Proof. Define

$$(4)\quad \begin{aligned} m(t) &= y(t)+y'(t)+\ldots+y^{(n-1)}(t),\\ m(o) &= y(o)+y'(o)+\ldots+y^{(n-1)}(o), \end{aligned}$$

then differentiating (4) and using the fact that $y^{(n)}(t) \leq a(t)\,m(t)$. $y^{(n-1)}(t)$ from (1) together with the facts that $y'(t)+y''(t)+\ldots+y^{(n-1)}(t) \leq m(t)$ and $y^{(n-1)}(t) \leq m(t)$ from (4) we see that the inequality

$$m'(t) \leq m(t) + a(t)\,m^{2}(t),$$

is satisfied for all t $\in$ I, which implies the estimation for m (t) suche that (see, [7])

$$(5)\quad m(t) \leq \frac{e^{t}\left[y(o)+y'(o)+\ldots+y^{(n-1)}(o)\right]}{1-\left[y(o)+y'(o)+\ldots+y^{(n-1)}(o)\right]\int_{o}^{t} e^{s}\, a(s)ds} = Q(t),$$

for all $t \in I$. Now, substituting this value of m(t) in (1) we have

$$(6) \qquad y^{(n)}(t) \leq a(t)Q(t)y^{(n-1)}(t),$$

for all $t \in I$. Dividing both sides of (6) by $y^{(n-1)}(t)$ and then integrating from o to t we obtain the estimate ofr $y^{(n-1)}(t)$ such that

$$(7) \qquad y^{(n-1)}(t) \leq y^{(n-1)}(o)\exp\left(\int_o^t a(s)Q(s)ds\right),$$

for all $t \in I$. The desired bound in (2) follows from (1), (5), and (7).

We note that, if we use the condition $y^{(n-1)}(t) \leq m(t)$ from (4) in place of condition (6) in (1), then in view of (5), the bound obtained in (2) reduces to

$$y^{(n)}(t) \leq a(t)Q^2(t),$$

for all $t \in I$.

Another interesting and useful n th order differential inequality is given in the following theorem.

THEOREM 2. Let $y(t), y'(t), \ldots, y^{(n)}(t)$, **and eache** $a_i(t)$ $(i = o,1,2,..n)$ **be real-valued nonnegative continuous functions defined on I expect that** $y^{(n-1)}(t)$ **is positive ofr all** $t \in I$, **for which the inequality**

$$(8) \quad y^{(n)}(t) \leq \sum_{i=1}^{n} a_i(t)\,(y(t)+y'(t)+\ldots+y^{(n-1)}(t))\,y^{(i-1)}(t)$$
$$+ a_o(t)(y(t)+y'(t)+\ldots+y^{(n-1)}(t)),$$

holds for all $t \in I$. **If**

$$1- \left[y(o)+y'(o)+\ldots+y^{(n-1)}(o)\right]\int_o^t \left(\sum_{i=1}^{n} a_i(s)\right)\exp\left(\int_o^s \left[1+a_o(t)\right]dt\right)ds > o$$

for all $t \in I$, **then**

$$(9) \qquad y^{(n)}(t) \leq \left(\sum_{i=1}^{n} a_i(t)\right)R^2(t) + a_o(t)R(t),$$

for all $t \in I$, where

$$(10)\ R(t) = \frac{[y(o)+y'(o)+\dots+y^{(n-1)}(o)]\exp\left(\int_o^t [1+a_o(s)]ds\right)}{1-[y(o)+y'(o)+\dots+y^{(n-1)}(o)]\int_o^t\left(\sum_{i=1}^{n} a_i(s)\right)\exp\left(\int_o^s[1+a_o(t)]dt\right)ds},$$

for all $t \in I$.

Proof. Define

$$(11)\quad \begin{aligned} m(t) &= y(t) + y'(t)+\dots+y^{(n-1)}(t), \\ m(o) &= y(o) + y'(o)+\dots+y^{(n-1)}(o), \end{aligned}$$

then differentiating (11) and using the fact that

$$y^{(n)}(t) \leq \sum_{i=1}^{n} a_i(t)\, m(t)\, y^{(i-1)}(t) + a_o(t)\, m(t)$$

from (8) together with the facts that

$$y'(t) + y''(t)+\dots+y^{(n-1)}(t) \leq m(t)$$

and

$$(12)\quad y^{(i-1)}(t) \leq m(t),\ (i=1,2,\dots,n)$$

from (11) we see that the inequality

$$m'(t) \leq [1 + a_o(t)]\, m(t) + \left(\sum_{i=1}^{n} a_i(t)\right) m^2(t)$$

is satisfied for all $t \in I$, which implies the estimation for $m(t)$ souch that

$$(13)\ m(t) \leq \frac{[y(o)+y'(o)+\dots+y^{(n-1)}(o)]\exp\left(\int_o^t[1+a_o(s)]ds\right)}{1-[y(o)+y'(o)+\dots+y^{(n-1)}(o)]\int_o^t\left(\sum_{i=1}^{n} a_i(s)\right)\exp\left(\int_o^s[1+a_o(t)]dt\right)ds} = R(t)$$

for all $t \in I$. The desired bound in (9) follows from (8), (12) and (13).

We next estalish the following n th order differential inequality which may be convenient in some applications.

THEOREM 3. **Let** $y(t)$, $y'(t), \ldots, y^{(n)}(t)$, $b(t)$, **and each** $a_i(t)$ $(i=1,2,\ldots,n)$ **be real-valued nonnegative continuous funcitions defined on** i, **for which the inequality**

$$(14)\quad y^{(n)}(t) \leq a_1(t)y(t) + a_2(t)y'(t) + \ldots + a_n(t)\, y^{(n-1)}(t) + b(t)\,(y(t) + y'(t) + \ldots + y^{(n-1)}(t)),$$

Holds for all $t \in I$. **Then**

$$(15)\quad y^{(n)}(t) \leq [\, b(t) + \sum_{i=1}^{n} a_i(t)][\, y(o) + y'(o) + \ldots + y^{(n-1)}(o)] .\exp\left(\int_o^t [1 + b(s) + \sum_{i=1}^{n} a_i(s)]\, ds\right),$$

for all $t \in I$.

The proof of this theorem follows by the similar argument as in the proof of Theorem 2 with suitable modifications, and we leave the datails to the reader.

3. **An Application.** In thes section we indicate a simple application of our Theorem 2 to obtain the bound on n th derivative of the solution of a class of n th order differential equation considered by Z. Opial [4] of the form (see, [1])

$$(16)\quad y^{(n)}(t) = \sum_{i=1}^{n} f_i(t, y(t), y'(t), \ldots, y^{(n-1)}(t))\, y^{(i-1)}(t) + k(t, y(t), y'(t), \ldots, y^{(n-1)}(t)),$$

$$y^{(p)}(o) = y_o^p \ , o \leq p \leq n-1,$$

where $y(t)$, $y'(t), \ldots, y^{(n)}(t)$, f_i $(i=1,2,\ldots,n)$ and k are the elments of R^n, an n-dimensional Euclidean space, and continuous on the respective domains of their definitions and y_o is a given positive constant.

Let /./ denote some convenient norm on R^n. Suppose that the functions f_i $(i=1,2,\ldots,n)$ and k in (16) satisfy

(17) $$|f_i(t,y(t),y'(t),\ldots,y^{(n-1)}(t)|$$

$$\leq a_i(t)[\,|y(t)|+|y'(t)|+\ldots+|y^{(n-1)}(t)|\,],$$

(18) $$|k(t,y(t),y'(t),\ldots,y^{(n-1)}(t))|$$

$$\leq a_0(t)[\,|y(t)|+|y'(t)|+\ldots+|y^{(n-1)}(t)|\,]$$

for all $t \in I$, where $a_i(t)$ $(i=o, 1, 2,\ldots n)$ are nonnegative continuous functions defined on I. Using (17) and (18) in (16) and applying Theorem 2 we have

(19) $$|y^{(n)}(t)| \leq (\sum_{i=1}^{n} a_i(t))Z^2(t) + a_0(t)Z(t),$$

for all $t \in I$, where

(20) $$Z(t) = \frac{[1+y_0+y_0^2+\ldots+y_0^{n-1}]\exp(\int_0^t [1+a_0(s)]ds)}{1-[1+y_0+y_0^2+\ldots+y_0^{n-1}]\int_0^t(\sum_{i=1}^{n} a_i(s))\exp(\int_0^s[1+a_0(t)]dt)ds};$$

(21) $$1-[1+y_0+y_0^2+y_0^{n-1}]\int_0^t(\sum_{i=1}^{n} a_i(s))\exp(\int_0^s[1+a_0(t)]dt)ds > 1,$$

for all $t \in I$. Further integrating (19) from o to t, n times one can very easily establish the bund on the solution $y(t)$ of (16).

Finally, we note tha the usefulness of the differential inequalities esblished in Theorems 1-3 becomes apparent if we consider

$y(o), y'(o), \ldots, y^{(n-1)}(o), a_i(t)$ $(i=o, 1, 2, \ldots,n)$ and $b(t)$ are known and $y(t)$, $y'(t),\ldots,y^{(n)}(t)$ are unknown functiones, i.e. the inequalities established in Theorems 1-3 gives us the bounds in terms of the known functions which majoriges $y^{(n)}(t)$ and consequently $y(t)$ after n times integration. In our future papers we wish to illustrate further applications of our inequalities to some other problems in the theory of n th order differential equations.

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