SERIES REPRESENTATIONS OF THE H-FUNCTION OF

TWO VARIABLES

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Sumary

Series representations for the H-function of two variables introduced earlier by Verma [10], are investigated when the poles of the integrand are assumed to be simple. Such representations seem to be non-existent in the literature. Since a number of density functions in a bivariate statistical probability distributions associated with an H-function of one variable are expressible in terms of H-function of two variables, the series-representtions obtained here would be useful in various probalems relating to statistical probability distributions.

Incidentally a special case of one of these series representations gives rise to the series-representation of the generalized Kampe de Feriet's function, defined and studied by Srivastava and Daoust [8] and whose convergence conditions are also discussed by them in a subsequent publication [9]. Consequently a general definition of the Srivastava and Daoust - function [8] has been obtained in terms of a double Mellin Barnes type integral.

1. Indtroducción

Agarwal (1965) has giben the series-representation of the meijer's G-function of two variables, when the poles of the integrand are assumed to be simple. Mathai and Saxena [5] obtained the expansion of the G-function of two variables in terms of the series of the generalized Zeta function and generalized Ps; -function, when the poles of the integrand differ in any manner.

In the present paper, authors have developed a series expansion for the H-function of two variables, introduced by Verma [10], following the notation of Saxena [12] a follows:

$$\text{(1.1)}\qquad H^{L,N,N,M,M}_{E,(A:C),F,(B:D)}\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}(e,\theta:\theta^1)\\ (\alpha,x);(c,\gamma)\\ (f,\phi:\phi_1)\\ (b,\beta);\ d,\delta)\end{matrix}\right]$$

$$=\left(\frac{1}{4\pi^2}\right)\int_{L_1}\int_{L_2}\chi_1\left(\begin{matrix}b_y;\ \alpha_j,\xi,\\ M,N;B,A\end{matrix}\right)\chi_2\left(\begin{matrix}d_j;c_j,\eta\\ M_1\ N_1;(D,C)\end{matrix}\right)\chi_3\left(\begin{matrix}1\text{-}f_j;e_j,\xi+\eta\\ -,L;\ F,E\end{matrix}\right)x\ y^{\eta}\,\delta\xi\,\delta\eta,$$

where

$$\chi_1\left(\begin{matrix}b_j;\alpha_j,\xi\\ M,\ N;B,A\end{matrix}\right)=\frac{\prod_1^M\Gamma(b_j-\beta_j\xi)\prod_1^N\Gamma(1-\alpha_j+\alpha_j\xi)}{\prod_{M+1}^{B}\Gamma(1-b_j+\beta_j\xi)\prod_{N+1}^{A}\Gamma(\alpha_j-\alpha_j\xi)}.$$

$$\chi_2\left(\begin{matrix}d_j,c_j,\eta\\ M_1,N_1;D,C\end{matrix}\right)=\frac{\prod_1^{M_1}\Gamma(d_j-\delta_j\eta)\prod_1^{N_1}\Gamma(1-c_j+\gamma_j\eta)}{\prod_{M_1+1}^{D}\Gamma(1-\delta_j+\delta_j\eta)\prod_{N_1+1}^{C}\Gamma(c_j-\gamma_j\eta)},$$

$$\chi_3\left(\begin{matrix}1-f_j;\ e_j,\xi+\eta\\ -,L;F,E\end{matrix}\right)=\frac{\prod_1^{L}\Gamma(1-e_j+\theta_j\xi+\theta_j\eta)}{\prod_{L+1}^{E}\Gamma(e_j-\theta_j\xi-\theta_j\eta)\prod_1^{F}\Gamma(1-f_j+\phi_j\xi+\phi_j'\eta)},$$

x, y are not zero and an empty product is interpreted as unity.

We can consider three different paths for L_1 and L_2

(I) L_1 and L_2 run from $-i\infty$ to $+i\infty$ in ξ and η planes respectively with indentations, if necessary, to ensure that the poles of $\Gamma(b_j-\beta_j\xi)$, $j=1,\ldots,M$, and $\Gamma(d_j-\delta_j\eta)$ are separated from the poles of $\Gamma(1-e_j+\Theta_j\xi+\Theta_j\eta)$, $\Gamma(1-\alpha_j+x_j\xi)$ and

$\Gamma(1-c_j+\gamma_j\eta)$. The integral (1.1) converges if :

$$\Psi_1 = \sum_{1}^{L} \theta_j - \sum_{L+1}^{E} \theta_j + \sum_{1}^{N} \alpha_j - \sum_{N+1}^{A} \alpha_j + \sum_{1}^{M} \beta_j - \sum_{M+1}^{B} \beta_j - \sum_{1}^{F} \phi_j > 0,$$

$$\Psi_2 = \sum_{1}^{L} \theta_j^1 - \sum_{1}^{E} \theta_j^1 - \sum_{1}^{F} \phi_j^1 + \sum_{1}^{N_1} \gamma_j - \sum_{N_1+1}^{C} \gamma_j + \sum_{1}^{M_1} \delta_j - \sum_{M_1+1}^{D} \delta_j > 0,$$

$$|\arg x| < \Psi_1 \pi / 2, \quad |\arg y| < \Psi_2 \pi / 2 .$$

(II) L_1 and L_2 are the loops starting and ending at $+\infty$, in the ξ and η planes respectively, and encircling all the poles of $\Gamma(b_j - \beta_j \xi)$ and $\Gamma(d_j - \delta_j \eta)$ once in the clockwise direction, but monte of the poles of

$\Gamma[(1-e_j + \Theta_j \xi + \Theta_j^1 \eta), \Gamma(1-a_j + \alpha_j \xi)$ and $\Gamma(1-c_j + \gamma_j \eta)$.

The integral (1.1) converges if:

$$\sum_{1}^{E} \theta_j - \sum_{1}^{F} \phi_j + \sum_{1}^{A} \alpha_j - \sum_{1}^{B} \beta_j < 0,$$

$$\sum_{1}^{E} \theta_j^1 - \sum_{1}^{F} \phi_j^1 + \sum_{1}^{C} \gamma_j - \sum_{1}^{D} \delta_j < 0, \ |x| < 1, \ |y| < 1.$$

(III) L_1 and L_2 are the loops in ξ and η planes respectively, starting and ending at $-\infty$ and encircling all the poles of

$\Gamma(1 - a_j + \alpha_j \xi)$, $(j = 1, \ldots, N)$, $\Gamma(1 - c_j + \gamma_j \eta)$, $(j = 1, \ldots, N_1)$.

and $\Gamma(1 - e_j + \Theta_j \xi + \Theta_j^1 \eta)$, $(j = 1, \ldots, L)$, once

in the positive direction, but none of the poles of $(b_j - \beta_j \xi)$, $(j = 1, \ldots, M)$,

and $\Gamma(\alpha_j \delta_j \eta)$, $(j = 1, \ldots, M_1)$.

The integral (1.1) convergers if

$$\sum_{1}^{E} \theta_j + \sum_{1}^{A} \alpha_j \geq 1, \sum_{1}^{E} \theta_j^1 + \sum_{1}^{C} \gamma_j \geq 1,$$

$$\sum_{1}^{E} \theta_j - \sum_{1}^{F} \phi_j + \sum \alpha_j - \sum_{1}^{B} \beta_j > 0,$$

$$\sum_{1}^{E} \theta_j^1 - \sum_{1}^{F} \phi_j^1 + \sum_{1}^{C} \gamma_j - \sum_{1}^{D} \delta_j > 0, |x| > 1, \quad |y| > 1 .$$

This function has also been defined and studied independently in a slightly variant from by Kalla and Munot [4], Gupta and Mittal [6], and Pathak [7], but in essence the function remains the same.

2. SERIES REPRESENTATION OF (1.1)

THEOREM 1. If the poles of the Gamma functions occurring in the integrand of (1.1) are simple and none of the poles coincide, then

$$(2.1) \qquad H^{L,\, N,\, N_1\, M,\, M_1}_{E,\,(A:C),\, F,\,(B:D)} \begin{matrix} x \\ \\ y \end{matrix} \left| \begin{matrix} (e, \theta : \theta^1) \\ (a, \alpha) ; (c, \gamma) \\ (f, \phi : \phi^1) \\ (b, \beta) ; (d, \delta) \end{matrix} \right.$$

$$= \sum_{h=1}^{M} \sum_{k=1}^{M_1} \sum_{P=0}^{\infty} \sum_{\lambda=0}^{\infty} \frac{\prod_{j \neq h}^{M} \Gamma\left[b_j - \beta_j (b_h + \rho)/\beta_h \right] \prod_{1}^{N} \Gamma\left[1 - \alpha_j + \alpha_j (b_h + \rho)/\beta_h \right]}{\prod_{M+1}^{B} \Gamma\left[1 - b_j + \beta_j (b_h + \rho)/\beta_h \right] \prod_{N+1}^{A} \Gamma\left[\alpha_j - \alpha_j (b_h + \rho)/\beta_h \right]}$$

$$\times \frac{\prod_{j \neq k}^{M_1} \Gamma\left[d_j - \delta_j (d_k + \lambda)/\delta_k \right] \prod_{1}^{N_1} \Gamma\left[1 - c_j + \gamma_j (d_k + \lambda)/\delta_k \right]}{\prod_{M_1+1}^{D} \Gamma\left[1 - d_j + \delta_j (d_k + \lambda)/\delta k \right] \prod_{N_1+1}^{C} \Gamma\left[c_j - \gamma_j (d_k + \lambda)/\delta_k \right]}$$

$$\times \frac{\prod_{1}^{L} \Gamma\left[1-e_j+\theta_j(b_h+\rho)/\beta_h+\theta_j^1(\delta_k+\lambda)/\delta_k\right]}{\prod_{L+1}^{E} \Gamma\left[e_j-\theta_j(b_h+\rho)/\beta_h-\theta_j^1(d_k+\lambda)/\delta_k\right] \prod_{1}^{F} \Gamma\left[1-f_j+\phi_j(b_h+P)/\beta_h+\phi_j^1(d_k+\lambda)/\delta_k\right]}$$

$$\times \frac{x^{(b_h+\rho)/\beta_h} \; y^{(d_k+\lambda(/\delta_k} \; (-1)^{\lambda+\rho}}{\rho! \; \lambda! \; \beta_h \; \delta_k},$$

provided tahat

$$(b\,h+\rho)\,\delta_k \sim (\delta_k+\lambda)\,\beta_h\,,\ (h=1,\ \dots,\ M\ ;\ k=1,\ \dots,\ M_1)\ ,$$
$$(\rho, \lambda = 0, 1, \dots),$$

$$\sum_1^E \theta_j + \sum_1^A \alpha_j - \sum_1^F \phi_j - \sum_1^B \beta_j < 0,$$

$$\sum_1^E \theta_j^1 + \sum_1^C \gamma_j - \sum_1^F \phi_j^1 - \sum_1^D \delta_j < 0, |x| < 1, |\xi| < 1.$$

PROOF. The proof of the theorem follows, by computing the residues at the poles of

$$\xi = (b_h + P)/\beta_h\,, (h = 1, \dots, M, P = 0, 1, \dots) \text{ and}$$
$$\eta = (d_k + \lambda)/\delta_k\,, (k = 1, \dots, M_1\,; = \gamma\ 0, 1, \dots),$$

and using the contour defined in (II).

In an analogous manner we can establish:

THEOREM 2. The sereis expansion of (1.1) can also be given in the form:

$$(2.2)\qquad H^{L,\,N,\,N_1,M,\,M_1}_{E,(A:C),\,F,(B:D)}\left[\begin{matrix} x \\ y\end{matrix}\,\middle|\,\begin{matrix}(e,\theta:\theta^1)\\(\alpha,\alpha);\ (c,\gamma)\\(f,\phi:\phi^1)\\(b,\beta);\ (d,\delta)\end{matrix}\right]$$

$$=\sum_{h=1}^{N}\sum_{k=1}^{N_1}\sum_{\rho=0}^{\infty}\sum_{\lambda=0}^{\infty}\frac{\prod_{1}^{M}\Gamma\left[b_j-\beta_j(\alpha_h-1+\rho)\,|\,\alpha_h\right]\prod_{\substack{1\\ j\neq h}}^{N}\Gamma\left[1-\alpha_j+\alpha_j(\alpha_j-1+\rho)|\alpha_h\right]}{\prod_{M+1}^{B}\Gamma\left[1-b_j+\beta_j(\alpha_h\rho)\,|\,\alpha_h\right]\prod_{N+1}^{A}\Gamma\left[\alpha_j-\alpha_j(\alpha_h-1+\rho)\,|\,\alpha_h\right]}$$

$$\times\frac{\prod_{1}^{M_1}\Gamma\left[\delta_j-\delta_j(c_k-1+\lambda)|\,\gamma_k\right]\prod_{\substack{1\\ j\neq k}}^{N_1}\Gamma\left[1-c_j+\gamma_j\,(c_k-1+\lambda)/\gamma_k\right]}{\prod_{M_1+1}^{D}\Gamma\left[1-d_j+\delta_j\,(c_k-1+\lambda)/\lambda_k\right]\prod_{N_1+1}^{C}\Gamma\left[c_j-\gamma_j\,(c_k-1+\gamma)/\gamma_k\right]}$$

$$\times\frac{\prod_{1}^{L}\Gamma\left[1-e_j+\theta_j(d_k-1+\rho)/\alpha_h+\theta_j^1\,(c_k-1+\lambda)\,/\gamma_k\right]}{\prod_{L+1}^{E}\Gamma\left[e_j-\theta_j(\alpha_h-1+\rho)/\alpha_h-\theta_j^1(c_k-1+\lambda)/\gamma_k\right]\prod_{1}^{F}\Gamma\left[1-f_j+\phi_j(\alpha_h-1+P)/\alpha_h+\phi_j^1(c_k-1+\lambda)\,/\gamma_k\right]}$$

$$\times\frac{x^{(\alpha_h-1+\rho)/\alpha_h}\;\;y^{(c_k-1+\lambda)/\gamma_k}}{\rho\,!\;\lambda\,!\;\;\alpha_h\,\gamma_k}$$

provided that

$$(\alpha-1+P)\,\gamma_k\sim(c_k-1+\lambda)\alpha_h\;,\ (h=1,\dots,N\,;k=1,\dots,N_1),$$

$$(P,\lambda=0,1,\dots)\;,$$

$$\sum_1^E\theta_j+\sum^A\alpha_j-\sum_1^F\phi_j-\sum_1^B\beta_j>0\;,\;\sum_1^E\theta_j^1+\sum_1^C\gamma_j+\sum_1^F\phi_j^1-\sum_1^D\delta_j>0\;,|x|>,|y|>1.$$

3. APPLICATIONS

The following special cases of (2.1) are worth mentioning.

(i) By taking all the positive real constants $\Theta_j, \Theta_j, \alpha_j, \gamma_j, \phi_j, \phi_j, \beta_j, \delta_j$ as unity, (2.1) gives the series representation of Meijer's G- Punction of two variables , obtained earlier by Agarwal [1].

(ii) Taking $L = E = A'$, $N = \dot{A} = B'$, $N_1 = C = B''$, $M = M_1 = 0$, $F = C'$, $B = D'$, $D = D''$, reemplazando replacing
$1\text{-}e_j$, $1\text{-}f_j$, $1\text{-}a_j$,$1\text{-}c_j$,$1\text{-}b_j$ and $1\text{-}d_j$
e_j , f_j , a_j ; c_j b, and d_j respectivamente, respectively,

$b_h = 0, (h = 1, \ldots\ldots, M)$, $d_k = 0$, $(k = 1, \ldots\ldots, M_1)$, y and

$\beta_h = 1$, $(h = 1, \ldots, M)$, $\delta_k = 1$, $(k = 1, \ldots, M_1)$ llegamos a we arrive

at the generalized Kampe' de Feriet's function introduced by Srivastava and Daoust [8].

(iii) Taking all the real constats in (2.1)

as unity, ($L = E = m$, $F = n$, $N = A = x$, $b_h = 0$, $d_k = 0$, $\beta_h = 1$,

$(h = 1 \ldots\ldots, M)$, $\delta_k = 1$ $(k = 1 \ldots\ldots, M_1)$, $N_1 = C = \mu^1$, $M = M_1 = 0$, $B = \nu$, $D = \nu^1$ and replacing $1\text{-}e_j$, $1\text{-}f_j$, $1a\text{-}_j$, $1\text{-}c_j$, $1b_j$ y $1\text{-}d_j$ y $e_j, f_j, \alpha_j, c_j, b_j$, d_j respectively,

(2.1) leads to the well-known Kampe' de Feriét - fuction [8]

(iv) On using the identity,

$$\lim_{y \to 0} H^{O,\, N,\, O,\, \dot{M},\, 1}_{O,\, (A:O),\, O,\, (B:1)} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (\alpha, x)\, j - \\ - \\ (b, \beta)\, j\, (1, 1) \end{matrix} \right] = H^{M,\, N}_{A,\, B} \left(x \middle| \begin{matrix} (a, \alpha) \\ (b, \beta) \end{matrix} \right),$$

we arrive at the series expansion of an H - function due to Braaksma [3] .

In a similar manner, a number of special cases of (2.2) can be exhibited which involve the various special functions of two variables scattered throughout the literature but for the sake of brevity they are not prsented here.

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