INTEGRAL EQUATIONS INVOLVING THE G-FUNCTION AS KERNEL

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Abstract

The present study deals with certain integral equation whose solution can be found by using the technique of L and L^{-1} operators. L and L^{-1} denote the Laplace transform and its inverse respectively. Later some special cases are derived from the main result. The obtained formulae seem to be of general interest.

1.—INTRODUCTION

In recent works of Fox [3], certain type of integral equations have been solved, by means of L and L^{-1} operators. The operators L and L^{-1} denote the Laplace transform and its inverse respectively. These solutions require a comprehensive tables [2] of Laplac transform and its inverse, which exist. The object of this paper is to determine the solution of an integral equation with G-function as Kernel, by L and L^{-1} operators.

The integral equation of the type

(1.1)

$$\int_0^\infty (x/t)^v \; G^{2,0}_{1,2}\left(xt \,\middle|\, \begin{matrix} a \\ b,\ c \end{matrix}\right) f(t)\, dt = g(x)\,, \qquad x>0,$$

where g is given and f is the unknown function to be determined, whose solution will be developed here. In (1.1),

$G^{2,0}_{1,2}(x)$ [1, p. 207] denotes

(1.2)

$$G^{2,0}_{1,2}\left(x \,\middle|\, \begin{matrix} a \\ b,\ c \end{matrix}\right) = (2\pi i)^{-1} \int_C M_{2,1}(s)\, x^{-s}\, ds ,$$

where

(1.3)

$$M_{2,1}(s) = \Gamma(b+s)\,\Gamma(c+s)\ \{\Gamma(a+s)\}^{-1} ,$$

(1.4)

$$\mathcal{M}\left[G^{2,0}_{1,2}(x) \right] = M_{2,1}(s) ,$$

where $\mathcal{M}$ denotes the Mellin transform.

We recall the following theorem of Fox [3] which will be needed in establishing the solution.

THEOREM: If,

(i) $\alpha > 0,\ \tfrac{1}{2}\alpha + \beta > 0,\ t > 0,$

(ii) $s = \delta + i\mu$, δ and μ both real;
$F(s) \in \pounds\ (\tfrac{1}{2} - i\infty,\ \tfrac{1}{2} + i\infty)$,

then

(1.5)

$$L^{-1}\left\{ (2\pi i)^{-1} \int_C \Gamma(\alpha s + \beta)\ F(s)\ t^{-\alpha s - \beta}\, ds \right\}$$

$$= (2\pi i)^{-1} \int_C F(s)\, x^{-\alpha s+\beta-1}\, ds$$

where, for both integrals, the contour C may be the line $\delta = 1/2$, a line parallel to the imaginary axis in the complex s — plane.

Next, we recall the Mellin - Parseval theorem:

If,

(1.6)

$$M\,[h(u)] = H(s) \text{ and } M\,[f(u)] = F(s) ,$$

then

(1.7)

$$\int_0^\infty h(u)\quad f(u)\quad du = (2\pi i)^{-1} \int_C H(s)\quad F(1-s)\quad ds ,$$

where C is a suitable contour in s — plane and M denotes the Mellin transform.

2.—Solution of 1.1) as an equation

THEOREM: If,

(i) $b>0,\ c>0,\ a>-1/2$;
(ii) $f(x) \in £_2\,(0,\infty)$;
(iii) $s^{b+c-a}\,F(1-s) \in £\,(1/2-i\infty,\ 1/2+i\infty)$;
(iv) $F(1-s) \in £\,(1/2-i\infty,\ 1/2+i\infty)$ and
(v) $y^{-1/2}\,f(y) \in £\,(0,\ \infty)$, where f(y) is of bounded variation near the point $y=t$, then the solution of (1.1), as an integral equation of f(t) is:

(2.1)

$$f(t) = t^{\nu-b}\, L^{-1}\,\{\ x^{a-b}\, L\,[\ t^{a-c}\, L^{-1}\ \{\ x^{-c-\nu}\, g(x)\ \}\ ]\ \}.$$

PROOF. In order to solve (1.1), we first apply (1.7) to the left han side of (1.1). Then, we eliminate the three Gamma function factors from the integrand of the resulting equation by using L^{-1} and L operators. By applying the asymptotic expansion of the Gamma function [8] along the line $s = \frac{1}{2} + it$, for large $|t|$, it can be shown that we can use equation (1.5) to eliminate the Gamma function factor from the integrand by L^{-1} operator. Then, again we can apply L operator to eliminate the Gamma function factor from the denominator. Thus, we can arrive at the result (2.1).

3.—APPLICATIONS

(A) By taking

$$\nu = 0,\ a = \mu - + \tfrac{1}{2},\ b = 2\mu,\ c = 0,$$

our theorem leads to

COROLLARY: If,

(i) $\mu > 0$, $\mu - k > -1$;

(ii) $f(x) \in \mathfrak{L}_2(0, \infty)$;

(iii) $s^{\mu + k - \frac{1}{2}} F(1-s) \in \mathfrak{L}(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$;

(iv) $F(1-s) \in \mathfrak{L}(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, and

(v) $y^{\frac{1}{2}} f(y) \in \mathfrak{L}(0, \infty)$, where $f(y)$ is of bounded variation near the point $y = t$,

then

(3.1)

$$\int_0^\infty [(xt)^{\mu - \frac{1}{2}} e^{-\frac{1}{2}xt} W_{k,\mu}(xt)]\, f(t)\, dt = g(x),\ x > 0,$$

has solution:

(3.2)

$$f(t) = t^{-2\mu} L^{-1} \{ x^{-\mu - k + \frac{1}{2}} L [t^{\mu - k + \frac{1}{2}} L^{-1} \{ g(x) \}] \},$$

$W_{k,\mu}(z)$ being Whittaker's function [8,u.334]. This solution has been obtained recently by Fox [4,(12].

(B) Again, if we write $t^{-\nu} f(t) = f_1(t)$ and $t^{-\nu} g(t) = g_1(t)$, the equation (1.1) reduces to the integral equation:

(3.3)

$$\int_0^\infty G_{1,2}^{2,0}\left(xt \,\middle|\, \begin{matrix} a \\ b,c \end{matrix}\right) f_1(t)\, dt = g_1(x), \quad (x>0),$$

and whose solution can be written under the similar conditions of the main theorem as:

(3.4)

$$f_1(t) = t^{-b} L^{-1} \{ x^{a-b} L \; [t^{a-c} L^{-1} \; \{ x^{-c} g_1(x) \}] \}.$$

(C) With $\mu+k=\frac{1}{2}$ in (3.1), we find

(3.5)

$$L\,[f(t)] = g(x).$$

(3.6)

$$f(t) = L^{-1} \{g(x)\}.$$

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