A THEOREM ON INTEGRAL EQUATION

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1.—The object of this paper is to establish a theorem on functions of two variables which under suitable conditions satisfy the integral equation

$$f(x, y) =$$

$$x^{\lambda 1} \cdot {}^{\lambda 2} \int_0^\infty \int_0^\infty e^{a_1 x - b_1 u + a_2 y - b_2 v} \; J_{\lambda 1}^{M1} (x^{M1} u) \; J_{\lambda 2}^{M2} (y^{M2} v) \; f(u, v) \, du dv$$

where $J_\lambda^M(x)$ is Bessel - Maitland Function [1] defined by

$$J_\lambda^M(x) = \sum_{\alpha=0}^{\infty} \frac{(-x)^\alpha}{\alpha \, j \, \Gamma(1 + \lambda + M\alpha)} \quad ; \qquad (M > 0).$$

The transform involving above function as Kernel has been studied by R. Kumar [2].

It is interesting to note that this theorem leads to (i) a general expression for functions which are R_M, ν i.e. which satisfy the integral equation

$$f(x, y) =$$

$$\int_0^\infty \int_0^\infty \sqrt{xytz} \;\; J_M(xt) \; J_\nu(yz) \; f(t, z) \, dtdz, \; R(M, \nu) > -1$$

and (ii) a generalised result in two variables analogous to well known Tricomi's Theorem [3].

2.—*Theorem:* The function f (x, y) determined by

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{\nu_1 - i\infty}^{\nu_1 + i\infty} \int_{\nu_2 - i\infty}^{\nu_2 + i\infty} e^{px + qy} \varnothing (p, q)\, dp dq \tag{2.1}$$

satisfies the integral equation

$$f(x, y) =$$

$$x^{\lambda_1} y^{\lambda_2} \int_0^\infty \int_0^\infty e^{a_1 x - b_1 u + a_2 y - b_2 v} J_{\lambda_1}^{M_1} (x^{M_1} u)\, J_{\lambda_2}^{M_2} (y^{M_2} v)\, f(u, v)\, du dv$$

provided the functional relation (2.2)

$$\varnothing (p, q) = \frac{\varnothing \{ b_1 + (p - a_1)^{-M_1},\ b_2 + (q - a_2)^{-M_2} \}}{(p - a_1)^{1+\lambda_1}\ (q - a_2)^{1+\lambda_2}} \tag{2.3}$$

holds.

This theorem is valid for

(i) $f(u, v) = 0\,(u^{-1+\epsilon_1}, v^{-1+\epsilon_2})$

when u and v are small and $(\epsilon_1, \epsilon_2) > 0$.

(ii) $f(u, v) = 0\,(u^{\beta_1} \cdot e^{S_1 u}, v^{\beta_2} \cdot e^{S_2 v})$

when u and v are large.

(iii) $p \equiv \gamma_1 + i\eta_1$, $q = \gamma_2 + i\eta_2$,
$\gamma_1 > c_1$, $\gamma_2 > c_2$; $(M_1, M_2) > 0$;

$R(\lambda_1, \lambda_2) > -1$; $R(p - S_1) > 0$,
$R(p - S_2) > 0$, $R(b - S_1) > 0$

$R(b_2 - S_2) > 0$; $R(p - a_1) > 0$, $R(q - a_2) > 0$;
$R(c_1 - S_1) > 0$, $R(c_2 - S_2) > 0$.

If $R(S_1 - b_1) = R(S_2 - b_2) = 0$, the theorem holds for $(M_1, M_2) < 1$. For $M_1 = M_2 = 1$, the additional conditions $R(\lambda_1 - 2\beta_1) > 0$, $R(\lambda_2 - 2\beta_2) > 0$, should be satisfied.

Proof: Let $\varnothing(p, q)$ be Laplace transform of $f(x, y)$ so that

$$\varnothing(p, q) = \int_0^\infty \int e^{-px-qy} f(x, y)\, dxdy;$$

$R(p - S_1) > 0$, $R(q - S_2) > 0$.

Then if $f(x, y)$ satisfied (2.2) we have

$$\varnothing(p, q) = \int_0^\infty \int_0^\infty x^{\lambda_1} y^{\lambda_2} e^{-px-qy}\, dxdy \times$$

$$\int_0^\infty \int_0^\infty e^{+a_1x - b_2u + a_2y - b_2 - v} J_{\lambda_1}^{M_1}(x^{M_1} u)\, J_{\lambda_2}^{M_2}(y^{M_2} v)\, f(u, v)\, dudv$$

$$= \int_0^\infty \int_0^\infty e^{-b_1u - b_2v} f(u, v)\, dudv \times$$

$$\int_0^\infty \int_0^\infty x^{\lambda_1} . y^{\lambda_2}\, e^{-(p-a_1)x} . e^{-(q-a_2)y} J_{\lambda_1}^{M_1}(x^{M_1} u)\, J_{\lambda_2}^{M_2}(y^{M_2} v)\, dxdy \qquad \ldots (I_1)$$

$$= \int_0^\infty \int_0^\infty e^{-b_1u - b_2v} f(u, v)\, dudv \times \sum_{\alpha_1}^{\infty} \frac{(-u)^{\alpha 1}}{\alpha_1 j\, \Gamma(1+\lambda_1+M_1\alpha_1)} \times$$

$$\sum_{\alpha_2}^{\infty} \frac{(-v)^{\alpha 2}}{\alpha_2 j\, \Gamma(1+\lambda_2+M_2\alpha_2)} \times$$

$$\int_0^\infty \int_0^\infty x^{\lambda_1+M_1\alpha_1}\, y^{\lambda_2+M_2\alpha_2}\, e^{-(p-a_1)x}\, e^{-(q-a_2)y}\, dxdy \qquad \ldots (I_2)$$

$$= \int_0^\infty \int_0^\infty e^{-b_1u - b_2v} f(u_2\; v)\, dudv \times \sum_{\alpha_1}^{\infty} \frac{(-u)^{\alpha 1}}{\alpha_1 j\, (p-a_1)^{1+\lambda_1+M_1\alpha_1}} \cdot$$

$$\sum_{\alpha_2}^{\infty} \frac{(-u)^{\alpha 2}}{\alpha_2 j\, (q-a_2)^{1+\lambda_2+M_2\alpha_2}} \cdot$$

$\times$ dudv

$$= \frac{1}{(p-a_1)^{\lambda_1+1} \cdot (q-a_2)^{\lambda_2+1}} \times$$

$$\int_0^\infty \int_0^\infty e^{-b_1u-(p-a_1)^{-M1} \cdot u - b_2v - (q-a_2)^{-M2}v} f(u, v)\, dudv$$

$$= \frac{\varnothing\; \{b_1+(p-a_1)^{-M1} \;.\; b_2+(q-a_2)^{-M2}\}}{(p-a_1)^{\lambda 1+1} \;.\; (q-a_2)^{\lambda 2+2}}$$

and then by [4]

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{\mathcal{V}_1 - i\infty}^{\mathcal{V}_1 + i\infty} \int_{\mathcal{V}_2 - i\infty}^{\mathcal{V}_2 + i\infty} e^{px+qy} \varnothing (p, q) \, dp \, dq$$

provided $p \equiv$

$\mathcal{V}_1 + i\eta_1$, $q \equiv \mathcal{V}_2 + i\eta_2$,
$\mathcal{V}_1 > c_1$, $\mathcal{V}_2 > c_2$,
$R(c_1 - s_1) > 0$, and $R(c_2 - s_2) > 0$.

It only remains to justifity the inversions in the steps I_1 and I_2.
We know that

$$\text{as } x \to \infty,\ J_\lambda^M(x) = 0 \left[x^{-k(a+\frac{1}{2})} \exp. \left\{ (Mx)^k \frac{\operatorname{Cos}(\eta k)}{Mk} \right\} \right]$$

$$k = \frac{1}{M+1} \text{ and as } x \to 0\ J_\lambda^M(x) = 0\ (1)$$

Therefore both integrals converge absolutely by virtue of De La Valle Poussin's Theorem [5], the inversion in the step I_1 is justified.

La inversion in the step I_2 is valid since
(i) each term

$$T_\alpha(x) \equiv \frac{(-x^M u)^\alpha}{\alpha j\, \Gamma(1 + \lambda + M\alpha)}$$

of the series is continuous.

(ii) $\Sigma\, |T\alpha\,(x)|$ is uniformly convergent in arbitrary large interval $(0, \alpha)$

and (iii)

$$\sum_{\alpha=0}^{\infty} \int_0^\infty x^\lambda e^{-(p-a)x} |T\alpha\,(x)| \, dx$$

converges to

$$(p-a)^{-\lambda-1} \exp. \{p-a)^{-M} u\} .$$

Example:

If we take

$$\varnothing (p, q) = \frac{\Gamma\nu_1}{(p-a_1)^{\nu 1}} x \frac{\Gamma\nu_2}{(p-a_2)^{\nu 2}} ; R (\nu_1, \nu_2) > 0$$

and

$$b_1 = a_1 , b_2 = a_2$$

then the functional relation (2.3) is satisfied provided.

$$M_1 = \frac{\lambda_1+1-\nu_1}{\nu_1} , \quad M_2 = \frac{\lambda_2+1-\nu_2}{\nu_2}$$

Therefore

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{r_1-i\infty}^{r_2+i\infty} \int_{r_2-i\infty}^{r_2+i\infty} e^{px+qy} \frac{\Gamma\nu_1}{(p-a_1)^{\nu 1}} . \frac{\Gamma\nu_2}{(q-a_2)^{\nu 2}} dpdq$$

$$= x^{\nu_1-1} e^{a_1 x} . y^{\nu_2-1} e^{a_2 y} ,$$

satisfies the integral equation (2.2) with $b_1 = a_1$, $b_2 = a_2$

and

$$M_1 = \frac{\lambda_1+1-\nu_1}{\nu_1} , \quad M_2 = \frac{\lambda_2+1-\nu_2}{\nu_2}$$

provided

$p = r_1+i\eta_1$, $q = r_2+i\eta_2$; $r_1>c_1$, $r_2>c_2$;
$M_1>0$, $M_2>0$,

$R\,(p-a_1,\ q-a_2) > 0$; $R\,(c_1-a_1,\ c_2-a_2) > 0$;
$R\,(2\nu_1,\ 2\nu_2) > 0$, $R\,(\lambda_1+1,\ \lambda_2+1) > 0$.

3.—Corollary. If f (x, y) is the function determined by

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{r_1-i\infty}^{r_1+i\infty} \int_{r_2-i\infty}^{r_2+i\infty} a^{+px+ay}\ \varnothing\,(p, q)\ dpdq \qquad (3.1)$$

and $\varnothing\,(p, q)$ satisfies the functional relation

$$\varnothing\,(p, q) = \frac{\varnothing\ \{a_1+(p_1-a_1)^{-1}\quad a_2+(q-a_2)^{-1}\}}{(p-a_1)^{v1+1}\quad (q-a_2)^{v2+2}}$$

then

$$e^{-a_1\frac{x^2}{2}\ -a_2\frac{y^2}{2}}\ .\ x^{-\nu_1+\frac{1}{2}}\ y^{-\nu_2+\frac{1}{2}}\ f\left(\frac{x^2}{2},\ \frac{y^2}{2}\right)$$

is $R\nu_1,\ \nu_2$

provided

(i) $f(x, y) = 0\ (x^{-1+\epsilon_1},\ y^{-1+\epsilon_2})$
when x and y are small $(\epsilon_1,\ \epsilon_2) > 0$.

(ii) $f(x, y) = 0\ (x^{\beta_1}\ e^{s_1x},\ y^{\beta_2}\ e^{s_2y})$
when x and y are large.

(iii) $p \equiv r_1+i\eta_1$, $q \equiv r_2+i\eta_2$; $r_1 > c_1$, $r_2 > c_2$; $R\,(p, q) > 0$;
$R\,(\nu_1-\nu_2) > -1$; $R\,(p_1-s_1,\ q-s_2) > 0$; $R\,(p-a_1,\ q-a_2) > 0$;
$R\,(s_1-a_1,\ s_2-a_2) < 0$; $R\,(c_1-s_1,\ c_2-s_2) > 0$.
If $R\,(s_1 - a_1,\ s_2 - a_2) = 0$, the additional condition
$R\,(\nu_1-2\beta_1,\ \nu_2-2\beta_2) > -\frac{1}{2}$ must be satisfied. For on putting
$M_1=1$, $M_2=1$, $a_1=b_2$ and $\lambda_1=\nu_1$, $\lambda_2=\nu_2$ (2.3) reducesto (3.2) and gives
$M_1=1$, $M_2=1$, $a_1=b_2$, $a_2=b_2$ and $\lambda_1=\nu_1$, $\lambda_2=\nu_2$ (2.3)
reducesto (3.2) and gives

$$f(x, y) = x^{\nu_1} y^{\nu_2} \int_0^\infty \int_0^\infty e^{a_1 x - a_1 u + a_2 y - a_2 v} J'_{\nu 1}(xu) J'_{\nu 2}(yv) f(u, v)\, du dv$$

or

$$x^{\frac{-\nu_1}{2}} y^{\frac{-\nu_2}{2}} e^{a_1 x - a_2 y} f(x, y)$$

$$\int_0^\infty \int_0^\infty J_{\nu 1}(2\sqrt{xu}) J_{\nu 2}(2\sqrt{yv})\, u^{-\nu_{1/2}} v^{-\nu_{2/2}} e^{-a_1 u - a_2 v} f(u, v)\, du dv$$

Replacing x by $x^2/2$, y by $y^2/2$, u by $u^2/2$ and v by $v^2/2$ we get

$$x^{-\nu_1 + \frac{1}{2}} y^{-\nu_2/2 + \frac{1}{2}} e^{-a_1 x^2/2 - a_2 y^2/2} f(\frac{x^2}{2}, \frac{y^2}{2})$$

$$= \int_0^\infty \int_0^\infty \sqrt{xyuv}\, J_{\nu 1}(xu) J_{\nu 2}(yv)$$

$$u^{-\nu_1 + \frac{1}{2}} v^{-\nu_2 + \frac{1}{2}} e^{+a_1 \frac{u^2}{2} + a_2 \frac{v^2}{2}} \times f(\frac{u^2}{2}, \frac{v^2}{2})\, du dv$$

i.e.

$$x^{-\nu_1 + \frac{1}{2}} y^{-\nu_2 + \frac{1}{2}} e^{-a_2 \frac{x^2}{2} - a_1 \frac{y^2}{2}} f(\frac{x^2}{2}, \frac{y^2}{2})$$

is $R_{\nu 1, \nu 2}$

On putting $a_1 = 0$, $a_2 = 0$, we get.

if f (x, y) is the function determined by

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{r_1 - i\infty}^{r_1 + i\infty} \int_{r_2 - i\infty}^{r_2 + i\infty} e^{px+qy} \varnothing (p, q) \, dpdq$$

and $\varnothing$ (p, q) satisfies he funcional relation

$$\varnothing (p, q) = p^{-\nu_1 - 1} q^{-\nu_2 - 1} \varnothing \left(\frac{1}{p}, \frac{1}{q}\right)$$

then under suitable conditions

$$x^{-\nu_1 + 1/2} y^{-\nu_2 + 1/2} f\left(\frac{x^2}{2}, \frac{y^2}{2}\right)$$

is $R_{\nu 1, \nu 2}$.

4.—From ∫ 2, the following theorem is evident

Si $\varnothing (p, q) \doteqdot f(u, v)$

Then

$$\frac{1}{(p - a_1)^{\lambda_1 + 1} (p - a_2)^{\lambda_2 + 1}}$$

$$\varnothing \{ b_1 + (p - a_1)^{-M1}, b_2 + (q - a_2)^{-M2} \}$$

$$\doteqdot\doteqdot x\lambda^1 y\lambda^2 \int_0^\infty \int_0^\infty e^{a_1 x - b_1 u + a_2 y - b_2 v} J^{M1}{}_{\lambda 1} (x^{M1} u) J^{M2}{}_{\lambda 2}(x^{M2} u) f(u, v) dud\text{v}$$

provided

(i) $((u, v) = 0 ({}^{-1+\epsilon_1}, v^{-1+\epsilon_2})$
when u and v are small

(ii) $f(u, v) = 0\,(u^{\beta_1} e^{s_1 u}, v^{\beta_2} e^{s_2 v})$
when u and v are large

(iii) $R(\lambda_1\ \lambda_2) > -1$; $R(p-a_1, q-a_2) > 0$; $R(s_1-b_1, s_2-b_2) < 0$, $(M_1, M_2) > 0$; $R(p-s_1, q-s_2) > 0$.

If $f(u, v) = 0\,(u^{\beta_1} e^{s_1 u}, v^{\beta_2} e^{s_2 v})$ and $R(s_1-b_1, s_2-b_2) = 0$ then theorem holds for $(M_1, M_2) < 1$.

It also holds for $M_1 = 1$, $M_2 = 1$ provided additional condition $R(\lambda_1-2\beta_1, \lambda_2-2\beta_2) > -\frac{1}{2}$ is satisfied. In particular if $a_1=a_2=b_1=b_2=0$ and $M_1=M_2=1$, we obtain Tricomi's theorem in two variables.

If $\varnothing(p, q) \doteqdot\doteqdot f(u, v)$, then under suitable conditions

$$\int_0^\infty \int_0^\infty \left(\frac{x}{u}\right)^{\lambda_{1/2}} \left(\frac{y}{v}\right)^{\lambda_{2/2}} J_{\lambda 1}(2\sqrt{xu})\, J_{\lambda 2}(2\sqrt{yv})\, f(u, v)\, du dv$$

5.—A number of integrals can be evaluated with the help of this theorem.

As an illustration let us take

$$\varnothing(p, q) = \frac{1}{p^{\nu_1+}\, q^{\nu_2+1}}$$

So that

$$f(x, y) = \frac{x^{\nu_1} \,.\, y^{\nu_2}}{\Gamma(\nu_1+1)\ \Gamma(\nu_2+1)}\ ;\ R\ \nu_1, \nu_2) > -1$$

and

$$\frac{1}{p^{\lambda_1+1}\ q^{\lambda_2+1}}\ \varnothing\,\{1+p^{-M1}, 1+q^{M-2}\}$$

$$= \frac{p^{M_1(\nu_1+1)-\lambda_1-1} \; q^{M_2(\nu_2+1)-\lambda_2-1}}{(1+p^{M_1\nu_1+1}) \; (1+q^{M_2\nu_2+1})}$$

Case I. Cuando $M_1=M_2=3$; $\nu_1=\nu_2=-\frac{1}{2}$ and $\lambda_1=\lambda_2=\frac{1}{2}$, we have

$$\frac{1}{p^{3/2} \; q^{3/2}} \; \varnothing \, \{1+p^{-3}, 1+q^{-3}\}$$

$$= \frac{1}{(1+p^3)^{1/2} \, (1+q^3)^{1/2}}$$

$$\doteqdot\doteqdot \frac{4}{9} \, \pi \sqrt{xy} \;\; J_{1/6}, {}_{-1/6} (x) \; .$$

$$J_{1/6}, {}_{-1/6} (y)$$

where

$$J_M \; y \; \nu \; (x) \; - \; \frac{x^{M+\nu}}{3^{M+\nu} \, \Gamma(M+1) \;\; \Gamma(\nu+1)} \; x \; O \; F_2 \; (M+1, \nu+1; \; -\frac{x^3}{27}).$$

Therefore

$$\int_0^\infty \int_0^\infty u^{-1/2} \; v^{-1/2} \; J^3_{1/2} \; (x^3 \; u) \; J^3_{1/2} \; (y^3 \; v) \;\; e^{-u-v} \, dudv$$

$$= \frac{4\pi^2}{9} \; J_{1/6}, {}_{-1/6} (x) \; .$$

$$J_{1/6}, {}_{-1/6} (y)$$

Case II. When $M_1=M_2=3$; $\nu_1=\nu_2=-\frac{1}{2}$ and $\lambda_1=\lambda_2=-\frac{1}{2}$

we have

$$p^{-1/2} q^{-1/2} \varnothing \ (1+p^{-3}, 1+q^{-3}) = \frac{pq}{(p^3+1)^{1/2} \ (q^3+1)^{1/2}}$$

$$\doteqdot\doteqdot \frac{4}{9} \pi \sqrt{xy} \ J_{-1/6}, _{-5/6} (x) \ . \ J_{-1/6}, _{-5/6} (y)$$

and therefore

$$\int_0^\infty \int_0^\infty u^{-1/2} v^{1/2-} \ e^{-u-v} \ J^3_{-1/2} (x^3 u) \ J^3_{-1/2} (y^3 v) \, du dv$$

$$= \frac{4\pi^2}{9} xy \ J_{1/6}, _{-5/6} (x) \ . \ J_{1/6}, _{-5/6} (y)$$

Case III. When $M_1 = M_2 = 2$; $\nu_1 = \nu_2 = 0$; $\lambda_1 = \lambda_2 = 1$

$$\frac{1}{p^2 q^2} \varnothing \ (1+\frac{1}{p^2}, 1+\frac{1}{q^2}) = \frac{1}{(1+p^2) \ (1+q^2)} \doteqdot\doteqdot \text{Sin } x \text{ Sin } y$$

and therefore

$$\int_0^\infty \int_0^\infty e^{-u-v} \ J^2_0 (x^2 u) \ J^2_0 (y^2 v) \, du \, dv = \frac{\text{Sin } x \text{ Sin } y}{xy}$$

Case IV. WhenM$_1 = M_2 = 2$; $\nu_1 = \nu_2 = 0$; $\lambda_1 = \lambda_2 = 0$

$$\frac{1}{pq} \varnothing \ (1+\frac{1}{p^2}, 1+\frac{1}{q^2}) = \frac{1}{(1+p^2) \ (1+q^2)} \doteqdot\doteqdot \text{Cos } x \ . \ \text{Cos } y$$

and therefore

$$\int_0^\infty \int_0^\infty e^{-u-v} \ J^2_0 (x^2 u) \ J^2_0 (y^2 v) \, du dv = \text{Cos } x \ . \ \text{Cos } y$$

Case V. When $M_1 = M_2 = 2$; $\nu_1 = \nu_2 = -\frac{1}{2}$; $\lambda_1 = \lambda_2 = 0$

$$\frac{1}{pq} \varnothing \left(1+\frac{1}{p^2},\ 1+\frac{1}{q^2}\right) = \frac{1}{(1+p^2)^{1/2}\ (1+q^2)^{1/2}} \doteqdot\doteqdot J_0(x)\ .\ J_0(y)$$

and therefore

$$\int_0^\infty \int_0^\infty e^{-u-v}\ J_0^2\ (x^2\ u)\ \ J_0^2\ (y^2\ v)\ dudv$$

$$= \pi\ J_0(x)\ .\ J_0(y).$$

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