EVALUATION OF FINITE INTEGRALS INVOLVING M-FUNCTION OF TWO VARIABLES BY USING E OPERATOF

By R. R. MAHAJAN and RAJENDRA K. SAXENA

ABSTRACT:

In the present paper we have evaluated two definite integrals involving the product of M-function of two variables and generalised hupergeometric function. The integrals have been evaluated by using the finite difference operator E.

(2) *Introduction*:- The M-function of two variables occuring in this pape: is due to D. P. Mourya [5]. It is defined and represented as follows:

(2.1)

$$M(x, y) = M\left[\begin{array}{c|c|c} \left[\begin{matrix} m_1, & n_1 \\ p_1-m_1, & q_1-n_1 \end{matrix}\right] & \{(ap_1; \alpha p_1, \alpha p_1)\}; \{(bq_1; \beta q_1, \beta q_1)\} & x \\ \left(\begin{matrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{matrix}\right) & \{(cp_2, \gamma p_2)\}; \{(dq_2, \delta q_2)\} & \\ \left(\begin{matrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{matrix}\right) & \{(ep_3, \epsilon p_3)\}; \{(fq_3, \rho q_3)\} & y \end{array}\right]$$

$$= \frac{1}{(2\pi 1)^2} \int_{L_1} \int_{L_2} m(\xi, \eta)\, x^{\xi} . y^{\eta}\, d\xi . d\eta$$

where $\{(a_p, \alpha_p)\}$ stands for $(a_1, \alpha_1), (a_2, \alpha_2), ----, (a_p, \alpha_p)$ and

(2.2)

$$m(\xi, \eta) = \frac{\prod_{j=1}^{m_1} \Gamma(1-\alpha_j+\alpha_j \xi+\alpha_j\eta) \prod_{j=1}^{n_1} \Gamma(b_j-\beta_j\xi-\beta_j\eta)}{\prod_{j=m_1+j}^{p_1} \Gamma(a_j-\alpha_j \xi-\alpha_j\eta) \prod_{j=n_j+1}^{q_1} \Gamma(j-b_j+\beta_j\xi+\beta_j\eta)} x$$

$$\times \frac{\prod_{j=1}^{m_2} \Gamma(1-c_j+\gamma_j\xi) \prod_{j=1}^{n_2} \Gamma(d_j-\delta_j\xi) \prod_{j=1}^{m_3} \Gamma(1-e_j+\epsilon_j\eta)}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j-\gamma_j\xi) \prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+\delta_j\xi) \prod_{j=m_3+1}^{p_3} \Gamma(e_j-\epsilon_j\eta)} \times$$

$$\times \frac{\prod_{j=1}^{n_3} \Gamma(f_j-\rho_j\eta)}{\prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+\rho_j\eta)}$$

where p_i, q_i, m_i and n_i $(i = 1, 2, 3)$ are non-negative integers such that $0 \leqslant m_i \leqslant p_i$, 0 $0 \leqslant n_i \leqslant q_i$. a_j, b_j, c_j, d_j, e_j and f_j are complex numbers and α_j, β_j, γ_j, δ_j, ε_j and ϱ_j are positive real numbers.

No pole of $\Gamma(1-a_j+\alpha_j\xi+\alpha_j\eta)$, $\Gamma(1-c_j+\gamma_j\xi)$ and $\Gamma(1-e_j+\varepsilon_j\eta)$ coincide with any pole of $\Gamma(b_j-\beta_j\xi-\beta_j\eta)$, $\Gamma(d_j-\delta_j\xi)$ and $\Gamma(f_j-\varrho_j\eta)$ respectively.

L_1, L_1 are suitable contours. x and y are not egual to zero and

$x^\xi = \exp\{\xi(\log|x|+i \arg x)\}$;

$y^\eta = \exp\{(\log|y|+i \arg y)\}$, in which $\log|x|$ and $\log|y|$ denote the natural logrithms of $|x|$ and $|y|$.

The integral on the right-hand side of (2.1) is convergent under the following set of conditions.

(2.3)

$$\mu_1 \equiv \left[\sum_{j=1}^{m_1} \alpha_j - \sum_{j=m_j+1}^{p_1} \alpha_j + \sum_{j=1}^{n_1} \beta_j - \sum_{j=n_j+1}^{q_1} \beta_j + \sum_{j=1}^{m_2} \gamma_j - \right.$$

$$\left. - \sum_{j=m_2+1}^{p_2} \gamma_j + \sum_{j=1}^{n_2} \delta_j - \sum_{j=n_2+1}^{q_2} \delta_j \right] > O$$

$$\mu_2 \equiv \left[\sum_{j=1}^{m_1} \alpha_j - \sum_{j=m_1+1}^{p_1} \alpha_j + \sum_{j=1}^{n_1} \beta_j - \sum_{j=n_1+1}^{q_1} \beta_j + \right.$$

$$\left. + \sum_{j=1}^{m_3} \epsilon_j - \sum_{j=m_3+1}^{p_3} \epsilon_j + \sum_{j=1}^{n_2} \delta_j - \sum_{j=n_2+1}^{q_2} \delta_j \right] > O$$

$$\mu_3 \equiv \left[\sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{q_2} \delta_j - \sum_{j=1}^{p_1} \alpha_j - \sum_{j=1}^{p_2} \gamma_j \right] > O$$

$$\mu_4 \quad \left[\sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{q_3} \rho_j - \sum_{j=1}^{p_1} \alpha_j - \sum_{j=1}^{p_3} \epsilon_j \right] > O$$

(v) $|\arg x| < \frac{1}{2} \mu_1 \pi$, $|\arg y| < \frac{1}{2} \mu_2 \pi$.

(3) NOTATION: In this paper we use the notation:

$$M \left[\begin{array}{c|c} \cdots\cdots\cdots & \cdots\cdots\cdots \\ \begin{matrix} m_2 & , & n_2 \\ p_2 - m_2 & , & q_2 - n_2 \end{matrix} & \{(c_{p_2}, \gamma_{p_2})\} ; \{(d_{q_2}, \delta_{q_2})\} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array} \right| \begin{matrix} x \\ \\ y \end{matrix} \Bigg]$$

to denote that the paramenters shown as - - - are the same as that of M(x, y) in (2.1).

The finite difference operator E used in the paper [4, p. 33 with w=1] is defined as follows:

$$E_a f(a) = f(a+1), E^n_a f(a) = f(a+n)$$

(4) INTEGRALS: We establish the following integrals.

First Integral:

(4.1)

$$\int_0^1 \eta^{\rho-1} (1-\eta)^{\beta-1} . M \left(x\eta^{\frac{m}{h}} . y \right) x . {}_uF_v \left(\begin{matrix} \alpha u \\ \beta v \end{matrix} ; Z\eta^t \right) . {}_2F_1 \left(\alpha . 2 . \beta ; 1-\eta \right) . d\eta .$$

$$= \Gamma(\beta) . \sum_{\mu=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\alpha_j)_\mu . Z^\mu}{\prod_{j=1}^{v} (\beta_j)_\mu \ (\mu)!} \times$$

$$\times M \left[\left(\begin{matrix} m_2+2 , & n_2 \\ p_2-m_2 , & q_2+2-n_2 \end{matrix} \right) \left| \begin{matrix} \cdots\cdots \\ (1-\rho-\mu t, \frac{m}{h}), \quad \{(dq_2, \delta q_2)\}, \\ (1-\rho-\mu t-\beta+\alpha+\nu, \frac{m}{h}), \quad (1-\rho-\mu t-\beta+\alpha, \frac{m}{h}), \\ \{(cp_2, \gamma p_2)\} ; (1-\rho-\mu t-\beta+\nu, \frac{m}{h}) \\ \cdots\cdots \end{matrix} \right| \begin{matrix} x \\ \\ y \end{matrix} \right]$$

provided Re $(\beta)>0$, $u \leqslant v$ (or $u=v+1$ and $|z|<1$), none of $\beta_1, \beta_2, ----, \beta_v$ is zero or negative integer.

$$\text{Re } (\varrho+\frac{m}{h}:d_j/\delta_j) > 0 \text{ for } j=1;2; ----; n_2$$

Re $(\varrho+\beta-\alpha-\nu)$ 0, t is a positive integer and conditions in (2.3) are satisfied.

Second Integral:

(4.2)

$$\int_{o}^{t} x^{\rho-1}(t-x)^{\delta-1} M(zx^{\lambda} \;.\; {}_{\mu}F_{\nu}\left(\begin{matrix}\alpha\mu\\ \beta\nu\end{matrix}; kx^{\nu}(t-x)^{\mu}\right) dx$$

$$= t^{\rho+\delta-1} \sum_{g=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\alpha_j)_g \; (k\, t^{\mu+\alpha})^g \; \Gamma(\delta+\mu g)}{\prod_{j=1}^{\nu} (\beta_j)_g \; (g)!} \; x$$

$$x\; M\left[\begin{array}{c|c|c} \cdots\cdots & \cdots\cdots & x \\ \begin{pmatrix} m_2+1 \;, & n_2 \\ p_2-m_2 \;, & q_2+1-n_2 \end{pmatrix} & \begin{matrix} (1-\rho-\nu g,\ \lambda), \quad \{(dq_2,\ \delta q_2)\}, \\ \{(cp_2,\ \gamma p_2)\};\ (1-\rho-\nu g-\delta-\mu g,\ \lambda) \end{matrix} & \\ \cdots\cdots & \cdots\cdots & y \end{array}\right]$$

provided $u \leqslant v$ (or $u = v+1$ & $|kt^{\mu+2}| < 1$)

None of $\beta_1, ----, \beta_\nu$ is zero or negative integer.

Re $(\varrho + \lambda d_j/\delta_j) > 0$ for $j = 1,2, ----, n_2$.

$\lambda > 0$, Re $(\delta) > 0$ and conditions in (2.3) with x replaced by z are satisfied

Proof of (4.1):

We shall use the following integral in the proof of (4.1)

(4.3)

$$\int_{o}^{1} \eta^{\rho-1} (1-\eta)^{\beta-1} M(x\eta^{\frac{m}{h}} \;.\; y) \;.\; d\eta$$

$$= \Gamma(\beta) \;.\; M\left[\begin{array}{c|c|c} \cdots\cdots & \cdots\cdots & x \\ \begin{pmatrix} m_2+1 \;, & n_2 \\ p_2-m_2 \;, & q_2+1-n_2 \end{pmatrix} & \begin{matrix} (1-\rho,\ \frac{m}{h}),\ \{(dq_2,\ \delta q_2)\}, \\ \{(cp_2,\ \gamma p_2)\};\ (1-\rho-\beta,\ \frac{m}{h}) \end{matrix} & \\ \cdots\cdots & \cdots\cdots & y \end{array}\right]$$

which can easily be proved by expressing the M-function as Mellin-Barne's contour integral from (2.1) interchanging the order of integrations, evaluating the inner integral and again interpreting by using (2.1).

The conditions for convergence of (4.3) are incorporated in (4.1).

Multiplying (4.3) by

$$\frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j+\delta)\ \Gamma(\alpha)\ \Gamma(\nu) \qquad z^{\delta}}{\prod_{j=1}^{\nu} \Gamma(\beta_j+\delta)\ \Gamma(\beta)}$$

and operating both sides by exp ($E_{\varrho}^{t}\ E_{\delta}\ +\ E_{\alpha}\ E_{\beta}\ E_{\nu}$) we get:

(4.4)

$$\sum_{\mu=0}^{\infty}\ \sum_{g=0}^{\infty} \int_{0}^{1} \frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j+\delta+\mu)}{\prod_{j=1}^{\nu} \Gamma(\beta_j+\delta+\mu)} \qquad \eta^{\delta+\mu t-1}(1-\eta)^{\beta+g-1}$$

$$\frac{z^{\delta+\mu}\ \ \Gamma(\alpha+g)\ \Gamma(\nu+g)\ M(x\eta^{\frac{m}{h}}\ .\ y)\ d\eta}{\Gamma(\beta+g)\quad (g)!}$$

$$= \sum_{\mu=0}^{\infty}\ \sum_{g=0}^{\infty} \frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j+\delta+\mu)\ z^{\delta+\mu}\Gamma(\alpha+g)\ \Gamma(\nu+g)}{\prod_{j=1}^{\nu} \Gamma(\beta_j+\delta+\mu)\quad (\mu)!\quad (g)!}\ x$$

$$x\ M \left[\begin{array}{c|c|c} \cdots\cdots & \cdots\cdots & x \\ \begin{pmatrix} m_2+1\ , & n_2 \\ p_2-m_2\ , & q_2+1-n_2 \end{pmatrix} & \begin{array}{l} (1-\rho-\mu t,\ -),\qquad \{(dq_2,\ \delta q_2)\}, \\ \{(cp_2,\ \gamma p_2)\};\ (1-\rho-\mu t-\beta-g,\ \frac{m}{h}) \end{array} & \\ \cdots\cdots & \cdots\cdots & y \end{array} \right]$$

After interchanging the order of integration and summation and evaluating the summation, we get the left hand side of (4.4) as

(4.5)

$$z^{\delta}\int_{0}^{1}\eta^{\delta-1}(1-\eta)^{\beta-1}\,M\left(x\eta^{\frac{m}{h}},\,y\right)\;{}_{\mu}F_{\nu}\left(\begin{matrix}\alpha\mu+\delta\\ \beta\nu+\delta\end{matrix};\,z\eta^{t}\right)\times$$

$$\times\;{}_2F_1\,(\alpha,\,\nu,\,\beta;\,1-\eta)\,d\eta\;\frac{F(\nu)\prod_{j=1}^{\mu}F(\alpha_j+\delta)\;F(\alpha)}{\prod_{j=1}^{\nu}F(\beta_j+\delta)\,.\,F(\beta)}$$

Also replacing the M-function by double Mellin Barne's contour integral interchanging the orders of summation and integration and after little simplification, the right hand side of (4.4) becomes:

(4.6)

$$\frac{1}{(2\pi 1)^2}\,z^{\delta}\int_{L_1}\int_{L_2}m(\xi,\,\eta)\,x^{\xi}\,y^{\eta}\sum_{\mu=0}^{\infty}\frac{\prod_{j=1}^{\mu}\Gamma(\alpha_j+\delta+\mu)}{\prod_{j=1}^{\nu}\Gamma(\beta_j+\delta+\mu)}$$

$$\frac{\times\,z^{\mu}\,.\,\Gamma(\delta+\mu t+\frac{m}{h}\xi)\;\Gamma(\alpha)\;\Gamma(\nu)}{(\mu)!\;\;\Gamma(\rho+\mu\,.\,t+\beta+\frac{m}{h}\,.\,\xi}\times$$

$$\times\;{}_2F_1\,(\alpha,\,\nu,\,\rho+\mu t+\beta+\frac{m}{h}\xi;\,1)\;d\xi\,.\,d\eta$$

Now using Gauss theorem [8], interchanging the order of integration and summation and interpreting the result by using the right hand side of (4.4) further reduces to:

(4.7)

$$z^{\delta}\sum_{\mu=0}^{\infty}\frac{\Gamma(\nu)\prod_{j=1}^{\mu}\Gamma(\alpha_j+\delta+\mu)\;\;z^{\mu}\Gamma(\alpha)}{\prod_{j=1}^{\nu}\Gamma(\beta_j+\delta+\mu)\;\;(\mu)!}$$

$$\times\ M\left[\begin{array}{c|c|c} \cdots\cdots\cdots & \cdots\cdots\cdots & x \\ \begin{pmatrix} m_2+2, & n_2 \\ p_2-m_2, & q_2+2-n_2\end{pmatrix} & \begin{array}{ll} (1-\rho-\mu t, \frac{m}{h}), & \{(dq_2,\ \delta q_2)\}, \\ (1-\rho-\mu t-\beta+\alpha+\alpha, \frac{m}{h}), & (1-\rho-\mu t-\beta+\alpha, \frac{m}{h}), \\ \{(cp_2,\ \gamma p_2)\}; & (1-\rho-\mu t-\beta+\alpha, \frac{m}{h}) \end{array} & \\ \cdots\cdots\cdots & \cdots\cdots\cdots & y\end{array}\right]$$

Now from (4.5) & (4.7) after replacing $\alpha_j+\delta$ by α_j and $\beta_j+\delta$ by β_j the required result is obtained.

Proof of (4.2):

The following integral will be used in the proof of (4.2).

(4.8)

$$\int_0^t x^{\rho-1}\ (t-x)^{\delta-1}\ M\ (zx^{\lambda},\ y)\ dx$$

$$\Gamma(\delta)\ t^{\rho+\delta-1}\ M\left[\begin{array}{c|c|c} \cdots\cdots\cdots & \cdots\cdots\cdots & z \\ \begin{pmatrix} m_2+1, & n_2 \\ p_2-m_2, & q_2+1-n_2\end{pmatrix} & \begin{array}{ll} (1-\rho,\ \lambda), & \{(dq_2,\ \delta q_2)\}, \\ \{(cp_2,\ \gamma p_2)\}\ ; & (1-\rho-\delta,\ \lambda) \end{array} & \\ \cdots\cdots\cdots & \cdots\cdots\cdots & y\end{array}\right]$$

which can be proved by expressing the M-function as double Mellin Barne's type contour integral from (2.1) in the left hand side of (4.8) and interchanging the order of integrations and again interpreting the result by using (2.1).

Now multiplying both sides of (4.8) by

$$\frac{\prod_{j=1}^{\mu}\ \Gamma(\alpha_j+\delta)\ k^{\delta}}{\prod_{j=1}^{\nu}\ \Gamma(\beta_j+\delta)}$$

and operating both the sides by $\exp(E_{\varrho}^{\nu} E_{\sigma}^{\mu} E_{\delta})$ we get:

(4.9)

$$\sum_{g=0}^{\infty} \frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j+\delta+g) \quad k^{\delta+g}}{\prod_{j=1}^{\nu} \Gamma(\beta_j+\delta+g) \quad (g)!} \int_{o}^{t} x^{\rho+\nu g-1} (t-x)^{\sigma+\mu g-1} \; x \; M(zx^{\lambda}, y) \; dx$$

$$\sum_{g=0}^{\infty} \frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j+\delta+g) \quad k^{\delta+g}}{\prod_{j=1}^{\nu} \Gamma(\beta_j+\delta+g) \quad (g)!} \Gamma(\sigma+g\mu) \quad t^{2y+\rho+\mu g+\sigma-1} \; x$$

$$x \; M\left[\left(\begin{matrix} m_2+1 \;, & n_2 \\ p_2-m_2 \;, & q_2+1-n_2 \end{matrix}\right) \left| \begin{matrix} \cdots\cdots & \cdots\cdots \\ (1-\rho-\nu g, \lambda), & \{(dq_2, \delta q_2)\}, \\ \{(cp_2, \gamma p_2)\}; & (1-\rho-\nu g-\sigma-\mu g, \lambda) \\ \cdots\cdots & \cdots\cdots \end{matrix} \right| \begin{matrix} z \\ \\ \\ y \end{matrix} \right]$$

Interchanging the order of integration and summation and evaluating the summation the left hand side of (4.9) becomes:

$$k^{\delta} \frac{\prod_{j=1}^{\mu} \Gamma(\alpha_j+\delta)}{\prod^{\nu} \Gamma(\beta_j+\delta)} \int_{o}^{t} x^{\rho-1} (t-x)^{\sigma-1} \; M(zx^{\lambda}, y) \; x \; x \; {}_{\mu}F_{\nu} \left(\begin{matrix} \alpha\mu \\ \beta\nu \end{matrix} ; kx^{\nu} (t-x)^{\mu} \right) . dx$$

Now changing $\alpha_j+\delta$ into α_j and $\beta_j+\delta$ into β_j the required result is obtained.

The M-function of two variables is a very generalised function. On specialising the parameters suitably it can be reduced to R. P. Agrawal's

G-function [1], S-function due to Sharma B. L. [9], P-function due to Pathak [7], Appell's functions F_1, F_2, F_3, F_4 [2], H-function due to Munot and Kalla [6], Horn's function G_2 [3] etc. Hence a large number of results involving these functions can be obtained from the results occuring in this paper as their particulars cases.

REFERENCES:

1. AGRAWAL, R. P. "An extension of Meijer's G-function". Proc. of Nat. Inst. of Sciences. India. P. A. (6) vol. 30 pp. 536 - 546 (1965).
2. APPELL-ET KAMPE DE FERRIET. "Functions hypergeometrique et hyperspheriques polynomes d'Hermite Gauthier Villars Paris (1926).
3. HORN, J. Math. Ann-105 pp. 381 - 407 (1931).
4. MILNE, THOMSON. The calculus of finite differences Mac-Millan, London (1933).
5. MOURYA, D. P. Ph. D. thesis approved for ph. D. degree. University of Indore, Indore (1970).
6. MUNOT, P. C. & S. L. KALLA. On an extension of generalised function of two variables. Separata de la Revista Mathematics y Fisica Theorica. Vol. XXXI, 1971. Nos. 1 y 2. Faculted de C. Exacts Technologia. U.N.T.
7. PATHAK, R. S. Some results involving G and H functions. Bull Cal. Math. Society 62. (1970, 97 - 106).
8. RAINVELLE, E. D. Special functions. The MacMillan Company, New York (1960).
9. SHARMA, B. L. "On generalised functions of two variables I" Anna de pa. Soc. Sc. de Brux T 791 pp. 26-40 (1965).