

# AN INTEGRAL INVOLVING GENERALIZED RICE'S POLYNOMIALS AND A GENERALIZED FUNCTION OF N VARIABLES

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## ABSTRACT

The object of the present note is to evaluate an integral involving a product of a generalized function of  $n$  variables and generalized Rice's Polynomials. The results obtained are most general in nature. A large number of known and unknown results may follow as their particular case.

## 1. INTRODUCTION

Recently Saxena (p. 256) has defined a generalized function of  $n$  variables by means of  $n$  fold multiple integral of Mellin-Barnes type as

$$H [x_1, x_2, \dots, x_n] = H_{\substack{A, (M_n:N_n) \\ C, D, (P_n:Q_n)}} \left[ \begin{array}{c} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{array} \middle| \begin{array}{l} {}^* \\ ({}^cC, {}^rC) : ({}^dD, {}^sD) \\ \\ \\ ({}^{(n)}P_h, {}^{(n)}P_n) : ({}^{(n)}Q_n, {}^{(n)}Q_n) \end{array} \right]$$

$$(1.1.) \quad \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} F(s_1 + s_2 + \dots + s_n) \Theta(s_1, s_2, \dots, s_n) x_1^{-s_1} x_2^{-s_2} \dots x_n^{-s_n} ds_1 ds_2 \dots ds_n$$

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where

$$F(S) = F(s_1 + \dots + s_n) = \frac{\prod_{j=1}^A \Gamma(c_j - S\gamma_j)}{\prod_{i=A+1}^C \Gamma(1 - c_i + s\gamma_i) \prod_{j=1}^D \Gamma(d_j - S\delta_j)}$$

$$\Theta(s_1, s_2, \dots, s_n) = \prod_{r=1}^n \left\{ \frac{\prod_{j=1}^{N_r} \Gamma(1 - a_j + s_r \alpha_j^{(r)}) \prod_{j=1}^{M_r} \Gamma(b_j + s_r \beta_j^{(r)})}{\prod_{i=1+N_r}^{Q_r} \Gamma(a_i - s_r \alpha_i^{(r)}) \prod_{j=1+M_r}^{P_r} \Gamma(1 - b_j + s_r \beta_j^{(r)})} \right\}$$

Where  $x_j \neq 0$  ( $j = 1, 2, \dots, n$ ) and an empty product is interpreted as unity. Further  $A, C, D; M_1, \dots, M_n; N_1, \dots, N_n; P_1, \dots, P_n$  and  $Q_1, \dots, Q_n$  are all integers satisfying the inequalities  $Q \leq A \leq C, 1 \leq M_j \leq Q_j, 0 \leq N_j \leq P_j$  ( $J = 1, 2, \dots, n$ ),  $a_j$ 's,  $b_j$ 's,  $c_j$ 's and  $d_j$ ' are all complex numbers and  $\alpha$ 's,  $\beta$ 's,  $\gamma$ 's and  $\delta$ 's are all positive numbers. Also  $(x_n)$  denotes the sequence of  $n$  parameters  $(x_1, x_2, \dots, x_n)$  and the symbol  $(a_p, A_p)$  the sequence of  $p$ -ordered pairs  $(a_1, A_1), \dots, (a_p, A_p)$ . The sequence of parameters in the integrand of (1.1) are such that none of the poles coincide. The paths of integration are identified, if necessary, in such a manner that all the poles of  $\Gamma(c_j - S\gamma_j)$  ( $j = 1, 2, \dots, A$ ) and  $\Gamma(b_j - s_r \beta_j^{(r)})$  ( $j = 1, 2, \dots, M_r; r = 1, 2, \dots, n$ ) are separated from the poles of  $\Gamma(1 - a_j + s_r \alpha_j^{(r)})$  ( $J = 1, 2, \dots, N_r; r = 1, 2, \dots, n$ ).

From the asymptotic expansion of the gamma function, it readily follows that the integral in (1.1) converges if where

$$(1.2) \quad \lambda_i < 0, \mu_i > 0, |\arg x^i| < \frac{1}{2} \pi \mu^i \quad (i = 1, 2, \dots, n).$$

$$(1.3) \quad \lambda_i = \sum_{j=1}^C \gamma_j + \sum_{j=1}^{Q_i^{(r)}} \alpha_j - \sum_{j=1}^D \delta_j - \sum_{j=1}^{P_i^{(r)}} \beta_j$$

$$(1.4)$$

$$\mu_i = \sum_{j=1}^{M_i^{(r)}} \beta_j - \sum_{j=1+M_i}^{P_i^{(r)}} \beta_j + \sum_{j=1}^{N_i^{(r)}} \alpha_j - \sum_{j=1+N_i}^{Q_i^{(r)}} \alpha_j + \sum_{j=1}^A \gamma_j \sum_{j=1+A}^C \gamma_j - \sum_{j=1}^D \delta_j.$$

When  $n = 2$ , (1.1) reduces to H-function of two variable defined by Verma (p. 104), Munot and Kalla (p. 69) and various others, which

itself is a generalization of G-function of two variables introduced by Sharma (p. 28). On the other hand, for  $A = C = D = O$ , the generalized function of  $n$  variables breaks up into a product of  $n$  H-functions. We thus have the identity.

$$(1.5) \quad H_{\substack{O, (M_n: N_n) \\ O, O, (P_n: Q_n)}} \left[ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \middle| \begin{array}{c} \text{---} : \text{---} \\ (r) (r) (r) (r) \\ (a_j, \alpha_j) : (b_j, \beta_j) \end{array} \right] = \prod_{j=1}^n H_{\substack{M_j, N_j \\ P_j, Q_j}} \left[ \begin{array}{c} (j) (j) \\ (aP_j, \alpha P_j) \\ x_j \\ (j) (j) \\ (bQ_j, \beta Q_j) \end{array} \right]$$

The function defined by (1.1) generalizes nearly all the known special functions of  $n$ -variables and it also leaves the possibility of defining through this new H-symbol, a great many Special Functions, not so far recorded in the literature.

In the present note an integral involving generalized Rice's polynomials and H-function of  $n$ -variables has been evaluated and a few particular cases have been derived. The results obtained are most general in nature. By specializing the parameters of H-function of  $n$ -variables or of the generalized Rice's polynomials, defined by Khandekar p. 58), a large number of known and unknown results may follow as particular case of our results.

In what follows the symbol  $\Delta(k, \alpha)$  will be used to denote the set of  $k$ -parameters  $\frac{\alpha}{k}, \frac{\alpha+1}{k}, \dots, \frac{\alpha+k-1}{k}$ .

## 2. EVALUATION OF THE INTEGRAL

The main result to be established here is

$$\int_0^1 t^{p-1} (1-t)^{\delta-p-1} H_n^{(\alpha, \beta)}(\delta, p; vt) H[(1-t)^m u_1, (1-t)^m u_n] dt$$

$$= \Gamma(p) m^{-p} P_n^{(\alpha, \beta)}(1-2v)$$

(2.1)

$$\times H_{\substack{A+m, (M_n: N_n) \\ C+m, D+m, (P_n: Q_n)}} \left[ \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \middle| \begin{array}{c} \Delta(m, \delta-p), (c_j, \gamma_j) : (d_j, \delta_j), \Delta(m, \delta) \\ (r) (r) (r) (r) \\ (a_j, \alpha_j) : (b_j, \beta_j) \end{array} \right]$$

Proof: In the integrand of (2.1), we replace the H-function of  $n$ -variables by its Mellin-Barnes Contour integral (1.1) and change in the order

of integration, which is valid since the integrals involved in the process are absolutely convergent. Next we evaluate the inner integral with the help of a known result (Khandekar, p. 159) and interpret the results with the help of multiplication formula due to Gauss and Legendre (Erdelyi *et. al.*, p. 4) and (1.1) to obtain (2.1).

The result (2.1) is valid under the conditions

$$\begin{aligned} & (\operatorname{Re}[\delta + m \sum_{r=1}^n \min_{j=1, \dots, M_r} b_j^{(r)} / \beta_j^{(r)}] > \operatorname{Re} p > 0 \quad (j=1, 2, \dots, M_r; r=1, 2, \dots, n), \\ & \lambda_i < 0, \mu_i > 0, \quad |\arg \mu_i| < \frac{1}{2} \pi \quad (i=1, 2, \dots, n); \quad m \text{ being a positive integer.} \end{aligned}$$

### 3. PARTICULAR CASES

On restricting  $v = 1$  in (2.1), it reduces to

$$\int_0^1 t^{p-1} (1-t)^{\delta-p-1} H_n^{(\alpha, \beta)}(\delta, p, t) H[(1-t)^{m_{u_1}}, \dots, (1-t)^{m_{u_n}}] dt$$

(3.1)

$$= \frac{(-1)^n \Gamma(p) (1+\beta)_n}{n! m^p} H \begin{matrix} A+m, (M_n : N_n) \\ C+m, D+m, (P_n : Q_n) \end{matrix} \left[ \begin{matrix} u_1 | (c_j, \gamma_j), \Delta(m, \delta-p) : \\ u_2 | (d_j, \delta_j), \Delta(m, \delta) \\ \cdot \\ \cdot \\ \cdot \\ u_n | (a_j, \alpha_j) : (b_j, \beta_j) \end{matrix} \right]$$

valid under the conditions stated with (2.1).

On utilizing the identity (1.5) in the result (2.1), we obtain an integral involving a product of  $n$  H-functions and generalized Rice's polynomials.

In particular on taking  $n = 2$  in the result (2.1), we derive an integral involving H-function of two variables.

$$\int_0^1 t^{p-1} (1-t)^{\delta-p-1} H_n^{(\alpha, \beta)}(\delta, p; vt) H[(1-t)^{m_{u_1}}, (1-t)^{m_{u_2}}] dt$$

(3.2)

valid form

$$= \frac{\Gamma(p) P_n^{(\alpha, \beta)}(1-2v)}{m^p} H \begin{matrix} A+m, (M_1, M_2 : N_1, N_2) \\ C+m, D+m, (P_1, P_2 : Q_1, Q_2) \end{matrix} \left[ \begin{matrix} u_1 | \Delta(m, \delta-p), (c_j, \gamma_j) : \\ u_2 | (d_j, \delta_j), \Delta(m, \delta) \\ (aP_1, \alpha P_1), (aP_2, \alpha P_2) \\ (bQ_1, \beta Q_1), (bQ_2, \beta Q_2) \end{matrix} \right]$$

$$\operatorname{Re} [\delta + m (b_j/\beta_j + b_i/\beta_i)] > O \quad (j=1, 2, \dots, M_1; i=1, 2, \dots, M_2)$$

$$\lambda_i < O, \mu_i > O, |\arg u_i| < \frac{1}{2} \pi \mu_i \quad (i=1, 2); m \text{ being a positive.}$$

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